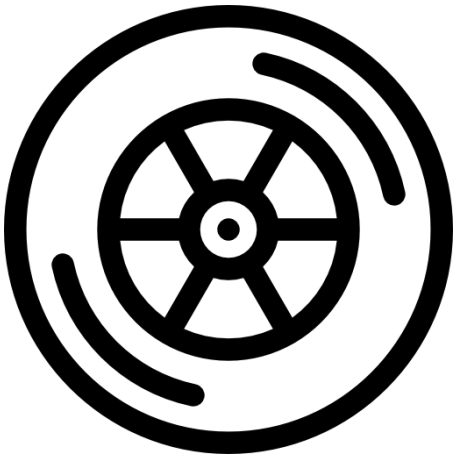
6.1 – Circular Motion



A-Level Physics:

0. Introduction
1. Measurements and their Errors
2. Particles and Radiation
3. Waves and Optics
4. Mechanics and Materials
5. Electricity
6. Further Mechanics and Thermal Physics
    - **6.1 Circular Motion**
    - 6.2 Simple Harmonic Motion
    - 6.3 Thermal Physics
7. Fields and their Consequences
8. Nuclear Physics
12. Turning Points in Physics

Objects moving in a curved path is just a special case of motion along curved path. The main reasons that we focus on circular motion is that the maths works out nicely and that there are countless examples of objects moving in paths which, if not exactly circular, can be approximated as such. The planets orbiting the Sun and the electron orbiting the atom are two examples. There are plenty of examples of circular motion in our everyday life: a CD spinning in a CD player, food in a microwave, a spinning football, a loop-the-loop at the fairground, a car going around a roundabout – these are all examples of circular motion.

There are three main parts to this booklet. In the first five sections, we develop the necessary tools to understand circular motion. Sections 6 to 8 then look at applying these tools to three different types of situation. There is only a small amount of new Physics here, but it can be applied to a vast array of different circumstances. My aim is that it will be easier to see the links between all the different types of problems if we break it down like this. Section 10 onwards develops our ideas of circular motion beyond the A-Level course and is intended for anyone looking for more to quench thirst for more circular motion.

There are 7 sets of Mastery Problems embedded within the main text. The problems focus on one specific skill that has been discussed in the previous section. The aim here is to isolate different types of problems to

ensure we understand the key points before proceeding. The mixed problems in Section 9 are intended to push you and require a greater degree of problem solving.

As ever, I am always looking to improve my resources so please let me know if there is anything that needs improving or any errors.

Mr Hynes

31/05/2020

Several Corrections Made

18/06/2020

Specification Reference

6.1 Circular Motion = AQA 3.6.1.1 Circular Motion

3.6.1.1 Circular motion (A-level only)

Content
Motion in a circular path at constant speed implies there is an acceleration and requires a centripetal force.
Magnitude of angular speed $\omega = \frac{v}{r} = 2\pi f$
Radian measure of angle.
Direction of angular velocity will not be considered.
Centripetal acceleration $a = \frac{v^2}{r} = \omega^2 r$
The derivation of the centripetal acceleration formula will not be examined.
Centripetal force $F = \frac{mv^2}{r} = m\omega^2 r$

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> ***Important Note**: Sections marked with a * and/or indented with a blue bar like this go beyond the course specification and are non-examinable. They build on the core content to help you build your understanding of circular motion. There are many parallels between circular and linear motion and these topics are designed to give the more advanced student a wider circular experience!*

1. Angles & Radians

If an object is moving along a circular path, it makes more sense to refer to its position on the circle using a radius r and an angle θ, as opposed to coordinates x and y. For circular motion, r will remain constant and θ will change, as shown in the diagram opposite.

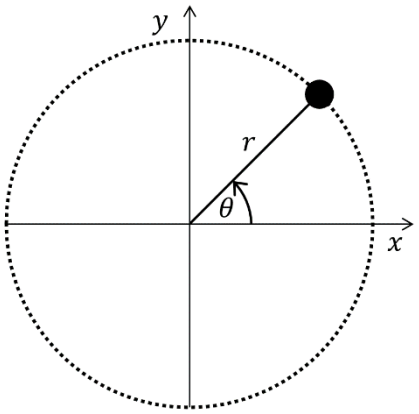


Figure 1.

We saw in the waves topic that it is sometimes more convenient to use **radians** as a unit to measure an angle. But where does the radian come from? In Figure 2, we see the angle θ in radians is defined as the ratio of the arc length s to the radius r, in other words

$$\theta = \frac{s}{r}$$

If $s = r$ then it follows that $\theta = 1$ rad. Therefore, 1 radian is the angle subtended at the centre of a circle by and arc equal in length to the radius. When $s = 2\pi r$ (the circumference of a circle of radius r), then $\theta = 2\pi$ radians = 360°. Hence

$$360° = 2\pi \text{ radians}$$

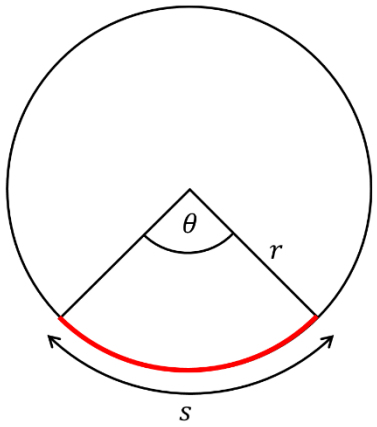


Figure 2

So why bother using radians then? Degrees can be useful – the number 360 has 24 divisors and is divisible by every number between 1 and 10, other than 7. This is a very useful property when discussing angles. However, radians can prove very useful when simplifying trigonometric expressions for small angles. The details are not necessary here, but it can be shown that

$$\sin\theta \approx \theta \text{ if } \theta \text{ is small and measured in radians.}$$

We will not be using that here, but it is always good to know why we do things. It is called the **small angle approximation,** should you want to go and find out more.

Example 1.1: A toy car moves around a circular track of radius 0.60 m. If the car travels 1.8 m along the track, what angles has the car moved through?

$$\theta = \frac{s}{r} = \frac{1.8}{0.6} = 3.0 \text{ rad} = 3.0 \times \frac{360}{2\pi} = 172°$$

Mastery Problems A – Angles & Radians

1. Without the use of a calculator, convert the following angles into radians in terms of π:

360°; 180°; 90°; 60°; 30°; 45°; 20°; 270°; 150°; 22.5°; 80°; 300°.

2. Without the use of a calculator, convert the following angles into degrees:

$$\frac{3\pi}{4};\quad \frac{5\pi}{6};\quad \frac{\pi}{10};\quad \frac{3\pi}{2};\quad \frac{11\pi}{6};\quad \frac{\pi}{3};\quad \frac{4\pi}{3};\quad \frac{\pi}{12};\quad \frac{\pi}{8};\quad \frac{4\pi}{9}$$

3. Express the following angles in radians:

35°; 82°; 53°; 71°; 44°.

4. Express the following angles in degrees:

2.86 rad; 4.21 rad; 1 rad; 3.47 rad; 3 rad.

2. Frequency & Time Period

Circular motion is a type of periodic motion. This means the motion repeats itself over a certain time interval. We can therefore use time period and frequency to describe circular motion:

Time period, T = time taken for one complete revolution [s].

Frequency, f = number of revolutions per second [s^{-1} = Hz]

where,

$$f = \frac{1}{T}$$

Note: a more commonly used unit for frequency for everyday objects is revolutions per minute, often abbreviated to rpm.

Example 2.1: A cyclist is racing around a circular velodrome at a constant speed. Their time for each lap is 23 s. How many revolutions of the track per minute does the cyclist do?

$$T = 23\ s \Rightarrow f = 1/T = 0.0435\ \text{Hz}$$

$$\text{Hence, } f = 0.0435 \times 60 = 2.6\ \text{rpm}$$

3. Angular Speed, ω

We can use the distance travelled and the time taken to calculate the speed of a journey for an object travelling in a straight line. This is known as the **linear speed**. If a car travels 1 mile in 1 minute, it travels at a speed of 1 mile/minute, which is equal to 60 mph.

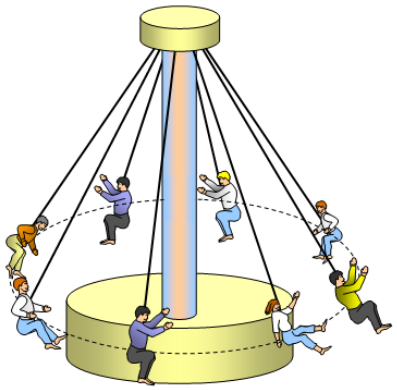


Figure 3

Similarly, we can use the angle travelled and time taken to calculate the **angular speed,** ω of an object. If the fairground ride in Figure 3 rotates through 2 radians in 1 s, we say the fairground ride has an angular speed `of 2 radians per second, or rad s^{-1}. We define ω as the rate of change of the angle:

$$\omega = \frac{\Delta\theta}{\Delta t}$$

ω = angular speed [rad s^{-1}]
$\Delta\theta$ = change in angle [rad]
Δt = change in time [s]

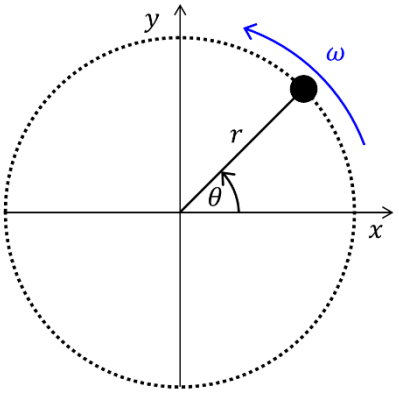


Figure 4

Note: the distinction between instantaneous speed and average speed that applied to linear speed also applies to angular speed. If the angular speed of an object is changing, then the formula above gives the average angular speed. The instantaneous angular speed would have to be found out using the derivative of θ with respect to time, but that is beyond the course scope.

Angular speed tells us how fast an object is rotating. In the next two sections, we will explore how angular speed is related to frequency and linear speed.

3.1 Angular Speed and Frequency

Frequency tells us how often an object rotates around a point and so does angular speed, so how are the two related? The answer lies in the units:

f → revolutions per second

ω → radians per second

To illustrate this, imagine a fairground ride like the one in Figure 3. It takes 5 s for a rider to complete a full rotation. Hence, T = 5 s, which implies f = 1/5 = 0.25 Hz. This means that the ride completes 0.25 revolutions per second. One revolution is 2π radians, so the radians per second, or angular speed, is

$$\omega = 2\pi \times 0.25 = \frac{\pi}{2} \text{ rad s}^{-1}$$

This demonstrates that f is related to ω by the following relation:

$$\boxed{\omega = 2\pi f}$$

Using $f = 1/T$, this also gives us:

$$\boxed{\omega = \frac{2\pi}{T}}$$

Mastery Problems B – Angular Speed

1. Complete the questions in the table by converting the units:

T / s	f / Hz	ω / rad s^{-1}	Revs. per min. (rpm)
0.50	(a)	(b)	(c)
(d)	(e)	3.0	(f)
(g)	(h)	(i)	3800
(j)	50	(k)	(l)
2700	(m)	(n)	(o)

2. Calculate the angular speeds and time periods of:
- **a.** a CD rotating at 100 rpm;
- **b.** a fan blade rotating at 5.2 rpm;
- **c.** a record player rotating at 33.3 rpm
- **d.** the Earth about its axis;
- **e.** the second hand on a clock;
- **f.** the minute hand on a clock;
- **g.** the hour hand on a clock.

3. Express 0.20 rad/minute in revs. per hour.

4. Express 50 degrees/minute in revs. per hour.

5. A car wheel turns 6.0 radians in 0.20 s. Calculate the angular speed of the wheel.

6. A car wheel turns 32° in 10 ms. Calculate the angular speed of the wheel.

7. The London Eye completes two revolutions in one hour. Calculate the:
- **a.** time period;
- **b.** frequency;
- **c.** angular speed.

3.2 Angular Speed and Linear Speed

Imagine a merry-go-round in a children's playpark – a firm favourite of the tiny human. If you're whizzing around at a constant angular speed, in which direction is your velocity? As shown in Figure 5, your direction is continuously changing when you are moving in a circle. Therefore, the direction of the velocity keeps changing even though you are moving at a constant speed.

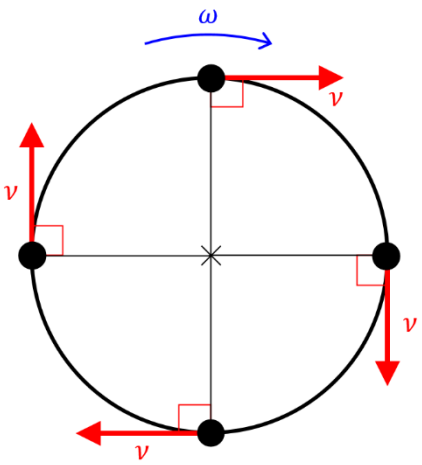
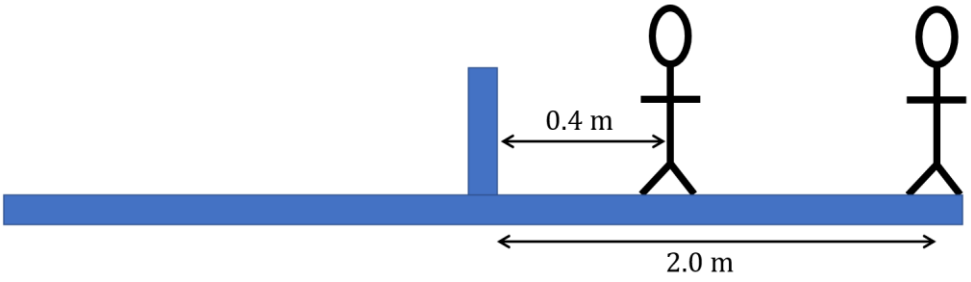


Figure 5

If you were to freeze this motion at any point, you would see that the direction of motion, and therefore the velocity, are both tangential to the circular path – hence it is often referred to as the **tangential velocity.** There is no component of the velocity in the radial direction at any instant as it is at 90° to the radial direction.

Now, say you're on this merry-go-round of radius 2.0 m with a friend. The ride is making a full rotation every second. You're on the very edge of the disc whereas your friend is only 0.4 m from the centre. They say the round-about travels slower closer to the centre – are they correct? Let's investigate.



The time period for the ride is 1 s so both you and your friend are travelling at the same *angular* speed. We can calculate the distance travelled for each person by calculating the circumference of the two circular paths. In one full rotation, at the edge of the ride you travel a distance of $2\pi \times 2.0 \approx 13$ m whereas your friend travels $2\pi \times 0.4 \approx 3$ m. You both completed one rotation in the same time of 1.0 s, therefore, you must have travelled faster if you travelled further in the same time. Your *linear* speed must be larger. Hence:

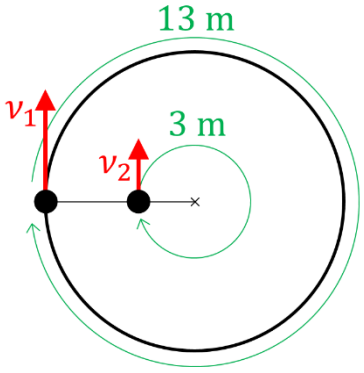


Figure 6

> ***For an object undergoing circular motion, the tangential velocity increases as you move away from the centre of rotation.***

Now let's derive the equation for this. Say a particle is moving in a circle with a constant angular speed ω. The particle travels through an angle θ from A to B in a time t, as shown in Figure 7. Since the particle is travelling at a constant angular speed, it must also be travelling at a constant tangential speed v, though its direction is changing so it is not travelling at a constant linear velocity. We know therefore that

$$v = \frac{\text{Arc length AB}}{t}$$

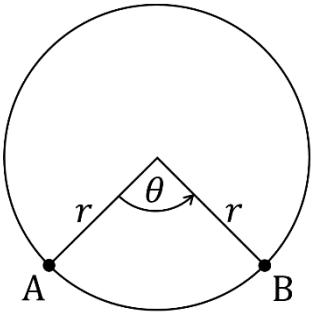


Figure 7

since the distance travelled from A to B is along the arc of the circle. Using $s = r\theta$, we know that

$$v = \frac{r\theta}{t}$$

And then using $\omega = \theta/t$ gives us our result

$$\boxed{v = \omega r}$$

ω	=	angular speed [rad s^{-1}]
v	=	tangential speed [m s^{-1}]
r	=	radius [m]

Notes:

- You may have noticed an apparent inconsistency with units in the above equation: if v has units of [m s^{-1}] and r has units of [m] then surely the units of ω should be

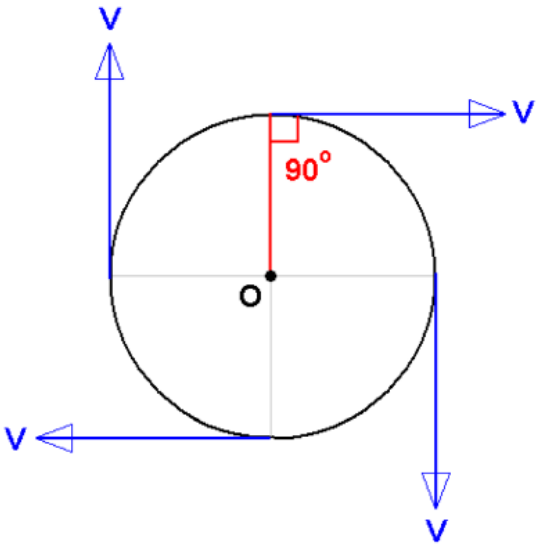
  $$[\text{m s}^{-1}]/[\text{m}] = [\text{s}^{-1}] \neq [\text{rad s}^{-1}]$$

  Good spot if you did! It's because *technically* speaking, radians have no dimensions. The definition of a radian comes from the formula $\theta = s/r$ where arc length and radius are both measured in metres. Therefore, technically speaking, an angle in radians is defined by a ratio of two lengths and therefore has no dimensions. In other words, don't worry about unit inconsistencies when it comes to radians!

- The velocity is always tangential to the circular path at a given instance. The speed may be constant but the velocity isn't!

- The angular speed needs to be in radians/second [rad s^{-1}] to use this equation.

- The tangential speed may be refered to as the linear speed or the instantaneous speed. They all mean the same thing.



Example 3.1 – a particle moves round a circle in 10 seconds at a constant speed of 15 m s^{-1}. Calculate the angular speed of the particle and the radius of the circle.

The particle rotates through an angle of 2π rad in $t = 10$ s therefore,

$$\omega = \theta/t = 2\pi/10 \approx 0.63 \text{ rad s}^{-1}$$

And then we can use $v = r\omega$ to give us

$$r = \frac{v}{\omega} = \frac{15}{0.628} \approx 24 \text{ m}$$

Mastery Problems C – Linear Speed

1. A disc is rotating about its centre with angular velocity ω. Point P is on the disc at a distance of d from the centre of the disk and has a linear speed of v.
- **a.** ω = 6.0 rad s^{-1}, d = 0.20 m, find v.
- **b.** v = 5.0 m s^{-1}, d = 0.40 m, find ω.
- **c.** v = 10 m s^{-1}, ω = 2.5 rad s^{-1}, find d.

2. A fairground Big Wheel carriage is 8.0 m from the centre of the wheel, which is rotating at 10 revs. per minute. Find the speed of the carriage.

3. My washing machine has a spin speed of 1200 rpm, and a drum radius of 25 cm. Calculate the linear speed of clothes at the edge of the drum.

4. A particle moves along a circular path of radius 4.0 m with an angular speed of 30 rad s^{-1}.
- **a.** Calculate the linear speed of the particle.
- **b.** Calculate the frequency in revs. per second.
- **c.** Calculate the time taken for one revolution.

5. Student A notices that the tip of the second hand appears to move faster on larger clocks than those on smaller clocks. Student B says this is not possible since both second hands will have the same angular speed on any clock. Explain Student A's observation.

6. In order to avoid supersonic effects, helicopter blades are designed so that the tip of the blade does not exceed the speed of sound, which is 340 m s^{-1} in air. The blades rotate at 260 rpm. Calculate the maximum length of the blades.

7. At a well, a bucket is attached to a thin rope which is round around a cylinder. The bucket is raised and lowered by turning the cylinder. The cylinder has a radius of 10 cm.
- **a.** If the cylinder is turned at 1 rev/second, at what speed will the bucket ascend?
- **b.** The well owner decides to change the cylinder so that when it is turned 1.0 rev/second, the bucket will ascend a 1.0 ms^{-1}. What is the radius of the new cylinder?
- **c.** State any assumptions you have made in solving this problem.

4. Centripetal Force

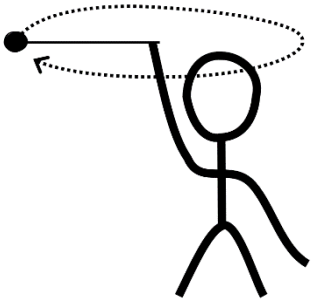


Figure 8

Consider swinging a ball on the end of a string around your head at a constant speed. Why does the ball keep moving in a circle? It should be obvious in this example that it is the tension in the string that keeps the ball moving in a circle. If you were to let go of the ball, it would fly off in a straight line at a tangent at the point where you let go. Figure 9 shows a ball on a string which if let go at point B, would continue in a straight line at a tangent to the circular path at B.

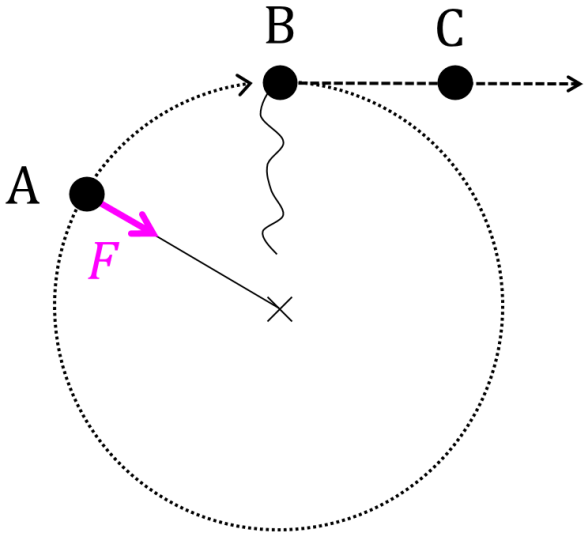


Figure 9 - Ball flying off at a tangent.

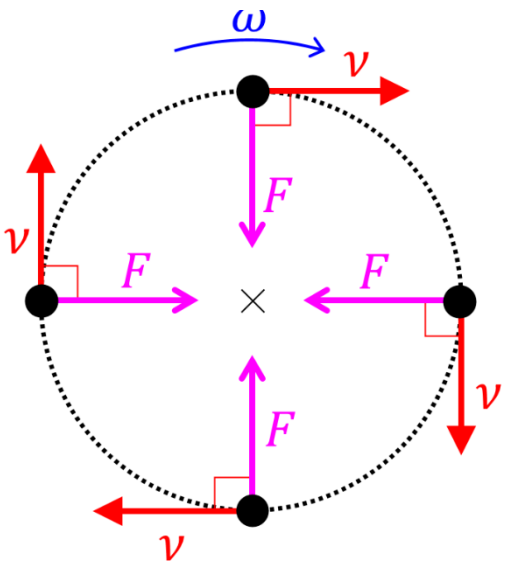


Figure 10

We call this force provided by the string that keeps the ball moving in a circle the **centripetal force**. We will see that this force always points to the centre of rotation, perpendicular to the path of the body at any point in its circular path. This makes sense; the ball wants to fly off in a straight line but the string pulls it back around in a circle.

But why is this force necessary? We discussed earlier how the velocity of a body moving in a circle is continually changing because the direction of motion is continually changing. Therefore, since the velocity of the body is continually changing, we can say the body is accelerating. Now, according to Newton's 2nd Law, there must be a resultant force acting on the body that is accelerating.

But, if the body is moving at a constant speed, at no point must there be a component of this force in the direction of motion of the ball. If there was, the ball would not travel at a constant speed. Hence, it follows that the force acting on the ball must always be perpendicular to the motion.

The resultant force of an object moving on a circular path must always be perpendicular to the tangential velocity and directed towards the centre of its path.

This argument about the requirement of a centripetal force for circular motion is essential to our understanding of circular motion. Re-read the last section and reproduce the argument in our own words on paper.

4.1 A Centrifugal Aside

> *"But hold up sir – I've been in a car and when we drive around a roundabout, I feel like I'm being pushed away from the centre of the circle, not towards the centre. I've heard the phrase 'centrifugal force' before, which refers to a force acting outwards from the centre. Shouldn't we call it a centrifugal force instead?"*

This is the most common source of confusement when studying circular motion – I remember being a very confused A-Level physicist myself! The answer comes from the fact that when you are inside the car, your point of view is distorted because you are accelerating. Hear me out.

Imagine you're on a train cruising at a constant speed. Everything seems normal – other than the slight vibration from the tracks, it is impossible to tell that you are moving. You decide now seems to be a good time to pop to the loo. Just as you get out of your seat and step into the aisle, the train brakes hard – what happens next? Indeed, you go flying forward, tripping over your feet and you faceplant the deck in truly spectacular fashion. What was this force that pulled you forwards? There was none! The train merely decelerated and because of Newton's 1st Law, you wanted to carry on moving forwards, so you did.

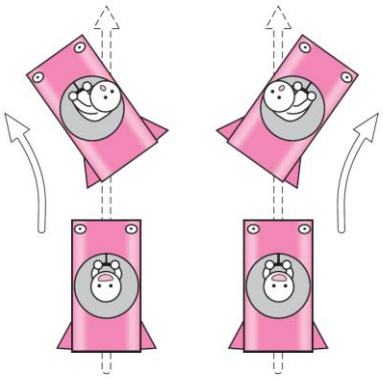


Figure 11

This is the same as the roundabout. As you go round the roundabout, the car changes direction (it accelerates) but you, because of Newton's 1st law, want to carry on in the straight line. This is best imagined when viewing the situation from above, as shown in Figure 11. As the car turns left, you continue moving in a straight line and move to the right until you hit the side of the car which provides the centripetal force to keep you going round in a circle. You weren't really pushed outwards, your reference frame merely accelerated – just like when the train brakes.

> **Here's another way of looking at it. We humans have an intuitive instinct to relate forces with motion and vice versa. We instinctively want to apply Newton's 1st Law and say that something will only start moving when a force acts on it. However, when we are in a frame of reference that is accelerating, Newton's 1st Law does not apply. In these frames of reference, it does appear that there is some centrifugal force pulling you outwards but stepping outside this frame of reference as in Figure 11 shows us this force does not really exist. They are only necessary to make Newton's 1st Law apply to accelerating reference frames. The centrifugal force is an example of a fictious force. Others include the Euler force and Coriolis force, the latter being responsible for the tendency of weather systems to deflect to the right in the Northern Hemisphere.*

Anyhow, all we need to know is:

Never ever use the term centrifugal force!

An object moving with circular motion has a force direction towards the centre called the centripetal force.

In order to quantify the size of the centripetal fore, we first need to investigate centripetal acceleration in the next section. The section after will then look at how we can calculate the size of the centripetal force and different examples of centripetal force.

5. Centripetal Acceleration

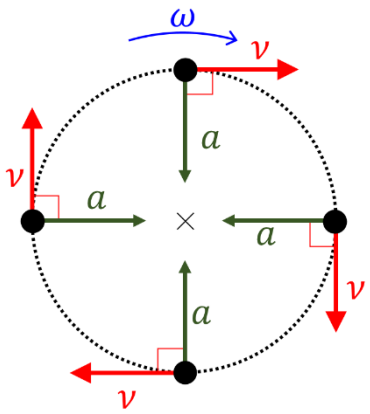


Figure 12

We saw in the last section that body undergoing circular motion is accelerating, since it is continually changing direction and hence velocity. Like the centripetal force, the direction of the **centripetal acceleration** is always towards the centre of the circular path and it can be shown (see section 5.2) that the magnitude of the acceleration is given by

$$a = \frac{v^2}{r}$$

a = centripetal acceleration [m s^{-2}]
v = tangential speed [m s^{-1}]
r = radius [m]

We can use this equation in conjunction with $v = \omega r$ to derive an equation for centripetal acceleration in terms of ω,

$$a = \omega^2 r$$

These two equations are the most important in the topic.

The acceleration of a particle moving at a constant speed v along a circular path of radius r has a magnitude of v^2/r and is directed towards the centre of the circle.

This is the most important statement in the topic of circular motion which encompasses the most important new physics.

Example 5.1 – a particle moves in a horizontal circle or radius 2.0 m at a constant speed of 3.0 m s^{-1}. Find the acceleration of the particle.

The particle is moving with uniform circular motion, therefore the acceleration is directed towards the centre of the circle and has a magnitude of

$$a = \frac{v^2}{r} = \frac{3^2}{2} = 4.5 \text{ m s}^{-2}$$

5.1 g-Forces

When you are in a car going round a bend, you can feel the effects of centripetal acceleration. It is often referred to as a 'g-force' in common parlance because centripetal accelerations are often expressed in units of the gravitational field strength, g. If a certain fairground ride had a centripetal acceleration of $2g$, this means that

$$a = 2g = 2 \times 9.81 \approx 20 \text{ m s}^{-2}$$

The amount of g-force a human can withstand depends on duration and direction of the force. Forces parallel to the length of the body can reduce blood circulation to the brain, causing unconsciousness. A centrifuge is the general name for a spinning device that utilises centripetal forces. In the video below, a rather large centrifuge is used to train fighter pilots to deal with g-forces. Video

*5.2 *Proof of $a = v^2/r$*

The proof of this equation is not required for this course – it is only for your interest! You can read about it on page 24 of Breithaupt or a slightly different proof here:

https://www.youtube.com/watch?v=TNX-Z6XR3gA

Mastery Problems D – Centripetal Acceleration

1. A particle moves with a constant speed of 3.0 m s^{-1} around a circular path of radius 50 cm. Find the centripetal acceleration of the particle.

2. A particle moves with a constant angular speed of 2.0 rad s^{-1} around a circular path of 1.5 m. Calculate the centripetal acceleration of the particle.

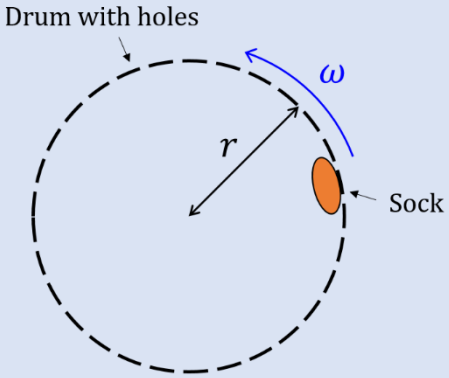
3. Rider in a bobsleigh experience high *g*-forces. Calculate the centripetal acceleration of a bobsleigh that is moving at 33 ms^{-1} and is on a corner of radius 38 m.

4. A record is revolving at 45 revs/minute. Points A, B and C lie on the record at 5.0 cm, 10 cm and 20 cm from the centre respectively. Find the accelerations of A, B and C.

5. An aircraft waiting to land at a busy airport is circling at a constant height at 500 km h^{-1}. The passengers experience an acceleration of 0.40*g* m s^{-2}. Find the radius of the circle.

6. An astronaut, as part of her training, is spun in a horizontal circle of radius 5.0 m. If she can withstand a maximum acceleration of 8*g*, what is the maximum angular speed at which she can remain conscious?

7. When a wet sock is spun in a washing machine, the drum rotates at high speed. Water in the sock escapes through the holes in the drum. In the diagram, the radius of the drum is 28 cm and the drum rotates at 860 rpm.



a. Calculate the speed of the rim of the drum.

b. Estimate the acceleration of the sock.

c. Explain why the water escapes through the holes in the drum.

8. The designer of a loop-the-loop ride in an amusement park wants to create an acceleration of 4*g* at the bottom of a loop of radius 8.0 m. At what speed should the car be moving to produce this acceleration?

We will now investigate three different types of circular motion problems in the following three sections:

i) motion in horizontal circles,
ii) motion in horizontal circles where the centripetal force is the horizontal component of a force e.g. conical pendula and banked turns,
iii) motion in vertical circles.

6. Motion in Horizontal Circles

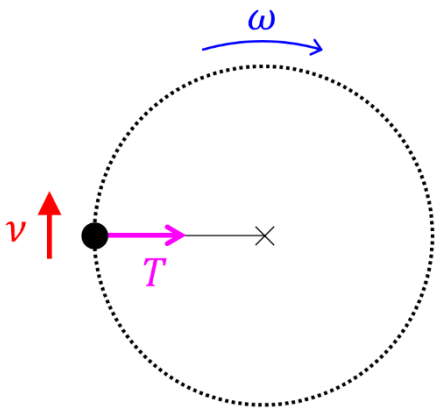
In this section, we will use the equation for centripetal acceleration to calculate centripetal force and apply this to objects moving in horizontal circles. We will begin by first continuing to look at a mass on a string and laying out the general method and results. We then proceed to look at other examples applying the same method.

Figure 13 - General Method

6.1 The Swinging Mass

Consider whizzing a mass m on a string around our head in deep space. You do it in such a way that the mass undergoes circular motion at a constant speed v. We're in deep space so the mass has no weight and therefore the only force is the centripetal force provided by the tension in the string. We'll see why later!

First thing as always is draw a nice diagram and label the forces acting on the ball. There is only the tension force T acting on the object. Then we continue as ever in mechanics and resolve forces. We choose to resolve forces in the direction of T, as this is also the direction of the centripetal acceleration a. Hence, by Newton's 2nd Law $F = ma$,



$$T = ma$$

We can substitute in the equation for centripetal acceleration to find

$$T = \frac{mv^2}{r}$$

And then using $v = \omega r$ gives us

$$T = m\omega^2 r$$

So, in general, we can say that the *resultant* centripetal force on an object undergoing circular motion is given by

$$\boxed{F = \frac{mv^2}{r} = m\omega^2 r}$$

Why am I stressing resultant here? Because in section 7, we will see problems where there is more than one force acting and so we shall have to find the resultant force in the radial direction. In every problem, I strongly advise you to resolve forces like any other mechanics problem but simply add on the equation for centripetal acceleration. There is no place for blindly applying an equation in A-Level Physics!

Before having a look at another example, let's analyse our result. Force increases with the mass and speed of the object moving in a circle. It makes sense that a faster and more massive object will need more force to keep continually changing its direction. For a given linear speed, the force decreases as the radius of the circle increase. This is because at a larger radius, the rate at which the direction of the object is changing is smaller – in other words, the corner in less tight.

We can relate this back to our discussion about cornering in a car from earlier – what sort of bend would you feel the largest force when going around a corner? It would be when you were going faster and when the bed was tighter (smaller radius).

Obviously, this assumes we are taking the corners at the same linear speed. If we were to take the corners at the same angular speed, $F = m\omega^2 r$ shows that a larger radius corner would have the greatest centripetal force. Cars don't have angular-speedometers though so this is less useful!

Here's the General Method for solving circular motion problems:

1. Resolve Forces
2. Use Newton's 2nd Law and $a = v^2/r$

Example 6.1 - A particle of mass 0.30 kg moves in a circular path with an angular velocity of 8.0 rad s^{-1} under the action a centripetal force of 2.0 N. Calculate the radius of the path.

The only force acting is the centripetal force. Resolving forces therefore gives us:

$$F = ma = \frac{mv^2}{r} = m\omega^2 r$$

We can rearrange this for r to get our answer:

$$r = \frac{F}{m\omega^2} = \frac{2}{0.3 \times 8^2} = 0.10 \text{ m}$$

Example 6.2 – Calculate the tension force required to keep a 1.5 kg mass in circular motion at the end of a 2.0 m piece of string with a frequency of 80 rpm.

Alarm bells should be going off as soon as you see a non-standard unit. Sort this out first before we forget!

$$f = 80 \text{ rpm} \quad \Rightarrow \quad f = 1.33 \text{ Hz}$$

The only force acting on the ball is the tension force T, which acts as the centripetal force. Resolving forces gives us

$$T = ma = \frac{mv^2}{r} = m\omega^2 r$$

We know that $\omega = 2\pi f$, hence

$$T = m(2\pi f)^2 r = 4\pi^2 \, mrf^2$$

Which gives us our answer

$$T = 4\pi^2 \times 1.5 \times 2.0 \times 1.33^2 = 210 \text{ N}$$

6.2 A Note on Energy

Is the total energy of the ball changing at all as it travels in a circle at a constant speed? Let's investigate:

> Kinetic – speed constant therefore kinetic store constant
>
> Gravitational – height of ball remains constant in horizontal circle therefore GPE remains constant.

So, in the absence of air resistance, the total energy of the ball remains constant. But surely the centripetal force acting on the ball does some work on the ball which changes the energy of the ball? Well, work done is given by $W = Fs$, where F is the force and s is the distance moved through so surely we would just multiply the centripetal force by the circumference of the circle to find the work done in one rotation? What's wrong there?

We've made a mistake in our definition of work done. It is the component of the force in the direction of the distance travelled. In other words, $W = Fs\cos\theta$, where θ is the angle between the force and the motion. Since the centripetal force is perpendicular to the motion, this means $W = 0$. Since work done is the energy transferred, this means that the centripetal force transfers no energy to the ball, in line with our previous statement that the total energy remains constant.

The centripetal force does no work.

6.3 Rounding the Bend

What provides the centripetal force for a car going round a roundabout?

The weight of the car acts downwards and the contact force pushes upwards on each tyre. The only other force acting is the friction between the tyres and the road. Friction is the force that opposes relative motion between two surfaces. The inertia of the car wants to continue in a straight line, so it is the friction between the tyres and the road that provides the necessary centripetal force to move the car with circular motion.

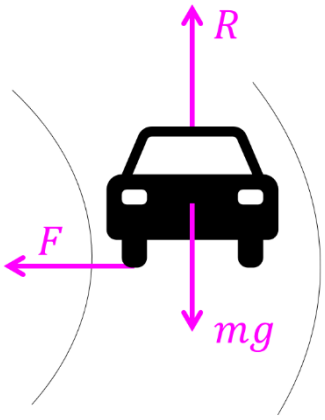


Figure 14 - Note: technically, there should be a frictional force and contact force on each tyre.

Friction provides the centripetal force for a car going around a bend.

The frictional force F acting on the car in the diagram is the only force acting in the horizontal plane, therefore when we resolve forces inwards, we arrive at

$$F = ma$$

from Newton's 2nd Law. Then we use our equation for centripetal acceleration to get

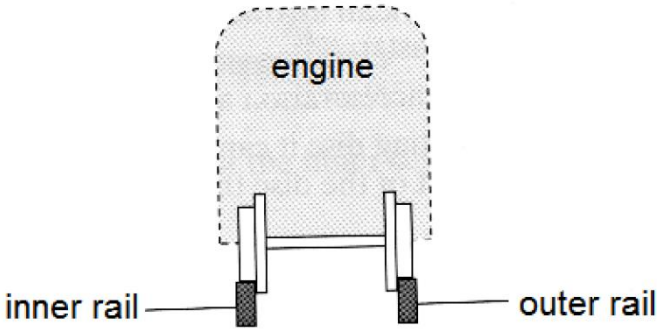
$$F = \frac{mv^2}{r} = m\omega^2 r$$

in the same way as the previous example. This tells us that a certain frictional force is required for a car of mass m, travelling at speed v, to round a bend of radius r. If the friction is not large enough for a certain bend, speed or mass, then the car will skid. In other words, driving in a certain car round a particular bend, the maximum speed you can take the

corner at is completely independent of driver skill and depends only on the maximum frictional force.

6.4 Take the A Train

Trains go round bends too, so we can solve problems with trains as well as cars – yes, the excitement is unbearable. Here's a diagram of a train on a track turning to the left. What provides the centripetal force here?



Most textbooks and some exam questions claim the centripetal force is provided by reaction force from the outer rail on the wheel flange. However, this is technically incorrect. The flanges are merely a safety mechanism in case the real mechanism fails.

In reality, the wheels of a train are slightly conical. This means the train tilts slightly going around a corner, causing the reaction force on the outside wheel to have a larger component directed towards the centre of the circle, thus providing the centripetal force. A more intuitive explanation involves the distance travelled and the size of the wheel, which is explained in videos linked below.

All we *need* to know is:

> ***A component of the reaction between the wheel and the track provides the centripetal force for a train going around a bend.***

Extra stuff on trains round bends if you're interested:

Video 1

Video 2

Video 3

Here's a quick diagram of a conical wheel, travelling in a straight line on the left diagram and rounding a left hand turn into the page on the right diagram. The horizontal components of the reaction force are balanced on the left but there is a resultant horizontal component on the right.

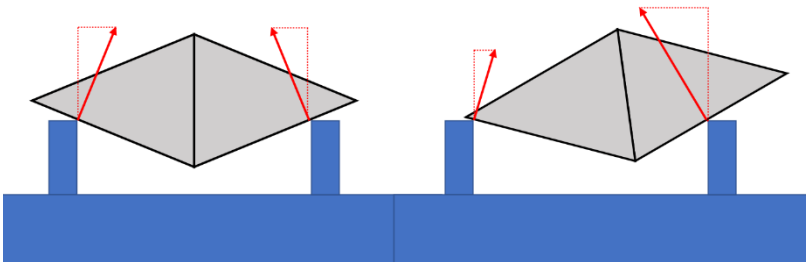
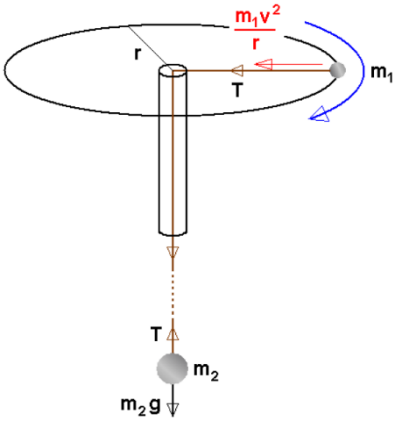


Figure 15 - Normal contact forces on conical wheel (forces not to scale).

6.5 Using Weight



A common setup for investigating circular motion and also for examination questions is shown in the diagram opposite. It consists of a hollow tube with a piece of string threaded through. There are masses m_1 and m_2 on either side of the string. The first mass, m_1, moves with circular motion in a horizontal plane. With the hollow tube example, a person may be holding the tube above their head and swinging the mass. Other arrangements in exam questions may have the top mass rotate on a smooth table attached to masses through a hole in the centre of the table – the physics is the same.

But why is this setup useful? Like our other examples, the centripetal force is supplied by the tension in the string. However, the tension is caused by the mass m_2 which is hanging vertical downwards. This allows us to keep the centripetal force at an easily measurable and constant value. Let's derive an expression for the time period of the setup.

Resolving the force on m_1, we have:

$$T = \frac{m_1 v^2}{r}$$

As T is the only horizontal force acting on m_1. Here, we have assumed m_1 is moving in a horizontal circle. We can then get this in terms of ω

$$\text{since } v = \omega r \quad \Rightarrow \quad T = m_1 \omega^2 r$$

and then in terms of the time period τ

$$\text{since } \omega = \frac{2\pi}{\tau} \quad \Rightarrow \quad T = \frac{4\pi^2 m_1 r}{\tau^2}$$

I used τ for time period as not to get confused with T for tension.

Okay, that's as far as we can get with m_1 – so let's resolve forces upwards on m_2. The same tension force that pulls the first mass in pulls the second mass up – hence we have

$$T = m_2 g$$

We can then set our two equations for T equal to each other to get

$$m_2 g = \frac{4\pi^2 m_1 r}{\tau^2}$$

And then rearrange for time period, τ

$$\tau = \sqrt{\frac{4\pi^2 m_1 r}{m_2 g}} = 2\pi \sqrt{\frac{m_1 r}{m_2 g}}$$

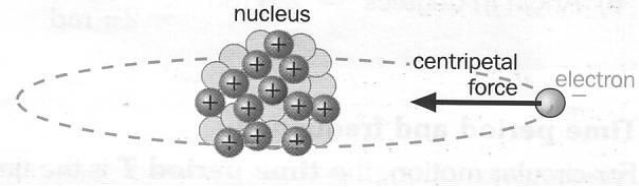
So, for a fixed radius of orbit r, the time period τ of the motion is constant – provided m_1 and m_2 are kept the same. In other words, the time period only depends on the radius.

See if you can use a straw or an empty biro, some string and two lumps of blue tack to replicate this experiment. By measuring the masses and measuring the time period of circles of different radius, could you use this apparatus to measure g?

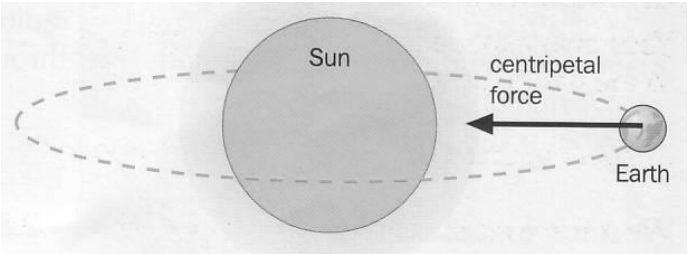
6.6 Other Examples of Centripetal Force

Later in the course, we will use the powerful concept of fields to investigate the gravitational, electric and magnetic forces. During this work, we will see how we can use our knowledge of circular motion to explain many concepts. The three key points for later learning will be:

1. We can model the electron moving around the nucleus as circular motion where the centripetal force is provided by the electric force.



2. We can model the orbits of planets as circular motion where the centripetal force is provided by the gravitational force.

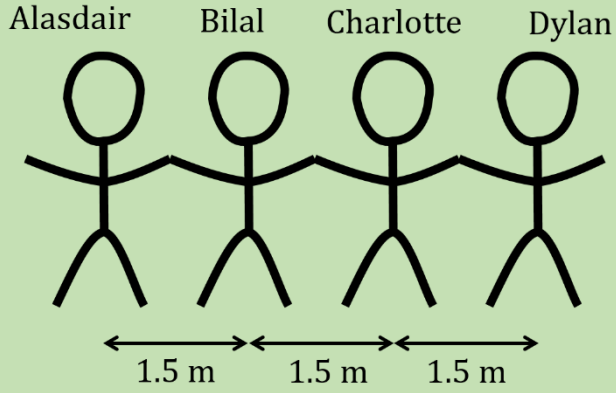


3. A charged particle in a magnetic field moves with circular motion. This follows from Fleming's left hand rule – the magnetic force is always perpendicular to the current (motion of charges) and so provides the centripetal force for the charge to move in a circular path.

I say this now as I believe it is important to try and make connections going forward as well as looking back.

6.7 More Worked Examples

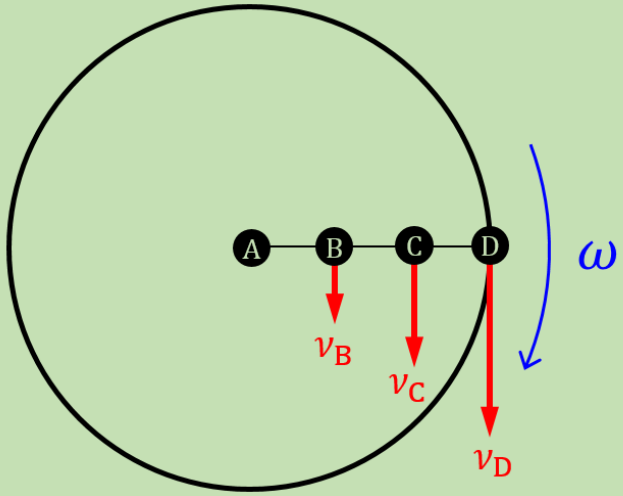
Example 6.3 – Alasdair, Bilal, Charlotte and Dylan are having a whale of a time skating. Their latest move involves them holding hands with their arms outstretched in a line as shown below.



They are all a similar size with a mass of 50 kg and the average span from left to right hand is 1.5 m. When they rotate, Alasdair rotates on the on the spot and the others move in a circle around him. If the angular speed of the group is 3.2 rad s^{-1}, calculate:

a. the linear speed of each skater,
b. the forces required between each pair to keep them in circular motion.

First things first, let's draw a nice diagram.



Lovely stuff. The diagram shows us that Dylan's linear speed must be the largest as he travels the furthest. In order to calculate each person's linear speed, we use $v = \omega r$ with subscripts to denote each person as follows:

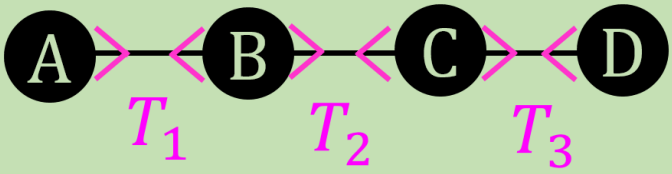
$$v_A = \omega r_A = 3.2 \times 0 = 0 \text{ m s}^{-1}$$

$$v_B = \omega r_B = 3.2 \times 1.5 = 4.8 \text{ m s}^{-1}$$

$$v_C = \omega r_C = 3.2 \times 2(1.5) = 9.6 \text{ m s}^{-1}$$

$$v_D = \omega r_D = 3.2 \times 3(1.5) = 14 \text{ m s}^{-1}$$

Now we need to work out the forces between each pair. Let's call the force between Alasdair and Bilal T_1, the force between Bilal and Charlotte T_2 and the force between Charlotte and Dylan T_3. This is shown in the diagram below.



We have used Newton's 3rd Law here, since we know the force Alasdair pulls Bilal with is equal and opposite to the force Bilal pulls on Alasdair with. The trick here is to analyse the motion of each person one at a time.

Let's start with Dylan because he only has one force acting on him, T_3. He moves in a circular of radius $r_D = 4.5$ m, therefore

$$T_3 = ma = m\omega^2 r_D = 50 \times 3.2^2 \times 4.5 = 2300 \text{ N}$$

Charlotte moves in a circle of radius $r_C = 3.0$ m but has two forces acting on her: T_2 pulls her towards the centre of the circle whilst T_3 pulls here away from the centre of the circle. Hence, resolving forces, we have

$$T_2 - T_3 = ma = m\omega^2 r_C$$

$$\Rightarrow \quad T_2 = m\omega^2 r_C + T_3$$

$$\Rightarrow \quad T_2 = (50 \times 3.2^2 \times 3.0) + (50 \times 3.2^2 \times 4.5) = 3800 \text{ N}$$

Finally, Bilal moves in a circle of radius $r_B = 1.5$ m with two forces acting on him: T_1 inwards and T_2 outwards. Therefore,

$$T_1 - T_2 = ma = m\omega^2 r_B$$

$$\Rightarrow \quad T_1 = m\omega^2 r_C + T_2$$

$$\Rightarrow T_1 = (50 \times 3.2^2 \times 1.5) + (50 \times 3.2^2 \times 3.0) + (50 \times 3.2^2 \times 4.5)$$

$$T_1 = 4600 \text{ N}$$

This shows us that the size of the force between each pair reduced as you move away from the centre of the circle. This demonstrates the importance of not just blindly applying $F = m\omega^2 r$, as that would have suggested the force increased. The reason is that Bilah is not just keeping himself moving in a circle with the force T_1, but he is keeping Charlotte and Dylan moving in a circle as well.

It should be noted that here we have assumed that the skaters are not using their skates to provide any frictional force towards the centre and we have modelled that skaters a particles. In reality, friction would provide some of the centripetal force and spatial extent of the skaters mass would have an effect – that's beyond A-Level Physics though!

Mastery Problems E – Horizontal Circles

1. A particle of mass 0.20 kg moves in a circular path with an angular speed of 5.0 rad s^{-1} under the action a centripetal force of 4.0 N. Calculate the radius of the path.

2. What force is required to cause a body of mass 3.0 g to move in circle of radius 2.0 m at a constant rate of 4.0 revs. per second?

3. A particle of mass 80 g rests at 16 cm from the centre of a turntable. If the maximum frictional force between the particle and the turntable is 0.72 N, what is the maximum angular speed at which the turntable could rotate without the particle slipping?

4. What is the force necessary to keep a mass of 0.80 kg revolving in a horizontal circle of radius 0.70 m with a period of 0.50 s?

5. A piece of string has a breaking strength of 300 N. Calculate the shortest time period a 2.0 kg mass can be swung round in a horizontal circle of radius 1.8 m.

6. We discussed how the string provides the centripetal force for a mass being swung around in a horizontal circle. Provided everything else remains constant, what happens to the tension in the string if the time period of rotation is halved?

7. A satellite of mass *m* is orbiting the Earth on a circular path at a height of 100 km above the surface. At this height, the acceleration due to gravity is 9.5 m s^{-2}. Take the radius of the Earth as 6400 km. For the satellite, calculate :

- **a.** the force exerted on it towards the centre of the Earth
- **b.** its tangential speed
- **c.** its angular speed
- **d.** the time taken to perform one orbit.

8. A ride at the funfair consists of cars which are made to move in a horizontal circle of radius 4.0 m, a t a rate of 0.40 revs/second. A girl of mass 45 kg is riding in the car. Find the horizontal and vertical components of the force exerted by the car on the girl.

7. Conical Pendula and Banked Turns

In every problem we have looked at so far, the force providing the centripetal force has always been directed into the centre of the circular path. This is not always the case.

7.1 More Mass on a String

If you've ever swung something around your head, you may have noticed we missed something when we discussed it in section 6.1. If you haven't tried it, now's your chance. Find something that will work a mass on a string system and spin it around your head. A wired computer mouse will do! You should notice that the string does not lie horizontally flat, but diagonally with the mass below the fixed end of the string. As you increase the angular speed for a given radius, the mass rises which decreases the angle between the string and the horizontal. However, no matter how fast you spin the mass, it will never be completely horizontal – why?

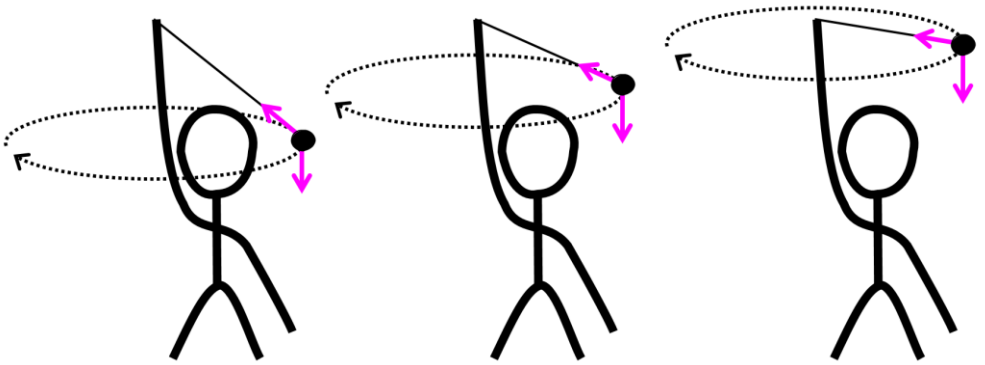


Figure 16 - Increasing angular speed from left to right.

When we looked at the mass on the string system in section 6.1, we simplified the problem and neglected the weight of the object. There are two forces acting on the mass:

i) its weight (mg) acting downwards,
ii) the tension in the string (T).

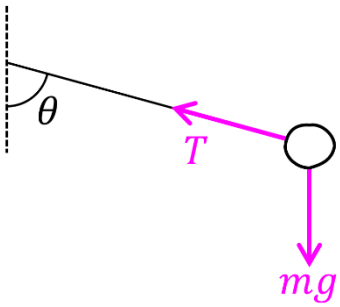


Figure 17

Since there are two forces acting on the mass, resolving forces vertically shows us that it is impossible for the tension to act horizontally. There needs to be a vertical component of the tension acting vertically upwards to counteract the weight pulling down. Hence, the mass moves in a horizontal circle below where the string is fixed, and the string sweeps out a cone.

Now, let's resolve forces to derive an equation for the angular speed. Note that we have defined θ as the angle between the vertical and the string due to convention. We resolve forces in the vertical plane to give us

$$T\cos\theta = mg$$

Then resolving horizontally and using $a = \omega^2 r$ gives

$$T\sin\theta = ma = m\omega^2 r$$

We see that it is only the horizontal component of the tension that provides the centripetal force. Turning the mathematical handle and dividing the horizontal equation by the vertical equation gives us

$$\frac{T\sin\theta}{T\cos\theta} = \frac{m\omega^2 r}{mg}$$

Everybody brought mass to the party (video) which leaves us with

$$\tan\theta = \frac{\omega^2 r}{g}$$

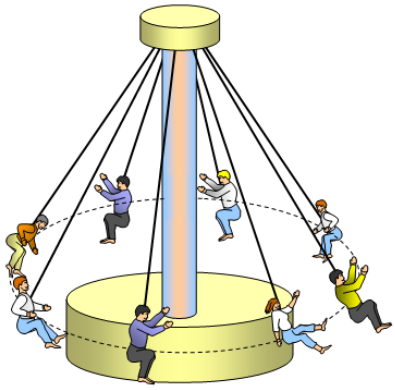


Figure 18

Cool. To take this further, let's consider a swinging chairs fairground ride shown in the Figure 18. The rope has a fixed length l which is related to θ and r by some simple trigonometry (Figure 19)

$$\sin\theta = \frac{r}{l} \quad \Rightarrow \quad r = l\sin\theta$$

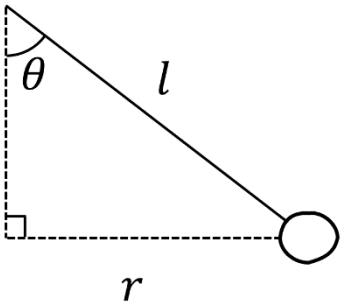


Figure 19

So, if we substitute this expression for r back into our equation, this gives us

$$\tan\theta = \frac{\omega^2 l\sin\theta}{g} \quad \Rightarrow \quad \frac{\sin\theta}{\cos\theta} = \frac{\omega^2 l\sin\theta}{g}$$

Then finally, a little rearranging gives us

$$\omega = \sqrt{\frac{g}{l\cos\theta}}$$

Tidy. What does that tell us? If you have ever seen swinging chair ride, you'll know that the ride starts off slow and then as the ride speeds up, the chairs swing outwards, just like our mass on the end of the string did before. Relating that to our expression, as ω increases, θ should increase. In our expression, g and l are both constants for a given ride – convince yourself that our expression agrees with our observation.

7.2 The Conical Pendulum

If you suspend a mass from a piece of string, it is referred to as a pendulum and after some time, it will come to rest at its point of equilibrium. If you displace the mass from this equilibrium point, it will oscillate about the point of equilibrium – oscillations will be the topic after circular motion. More interestingly for us, if we displace the mass and give it a tangential velocity, the mass will move in a horizontal circle. It is referred to as a conical pendulum as the string sweeps out a cone.

A conical pendulum is just a fancy name for a mass on a string moving in a horizontal circle – don't be fooled!

7.3 Banked Turns

https://www.youtube.com/watch?v=JodRZSFmOhY

Racing tracks back in the old days had banked turns to allow the cars to take corners at greater speed. As cars became faster, this practice became dangerous and was eventually banned. The best modern example of a banked turn is in the velodrome. Cycling in a velodrome is the most amazing experience – you go round in circles without having to turn at all! But why do banked turns allow you to corner at greater speeds?

Let's look at the forces acting on a car on banked turn shown in Figure 20. The weight of the car pulls it vertical downwards whilst the normal reaction acts perpendicular to the surface. We can see that the horizontal component of the normal reaction pushes the car towards the centre of its circular path and therefore contributes towards the centripetal force. This explains why banking allows for faster cornering. We have ignored friction for now – you will see why later.

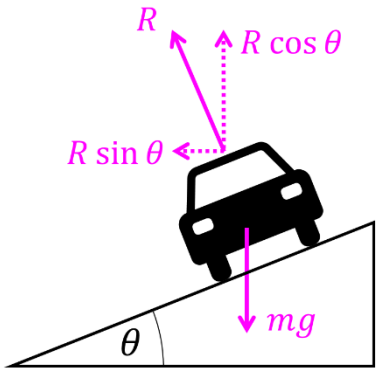


Figure 20 – technically, there is a reaction force on each wheel but we have just grouped them into one force here for ease of use.

If the car is cornering at a constant height on the banked turn, they are moving along a horizontal circular path. We first resolve vertically where there is no acceleration to get

$$R\cos\theta = mg$$

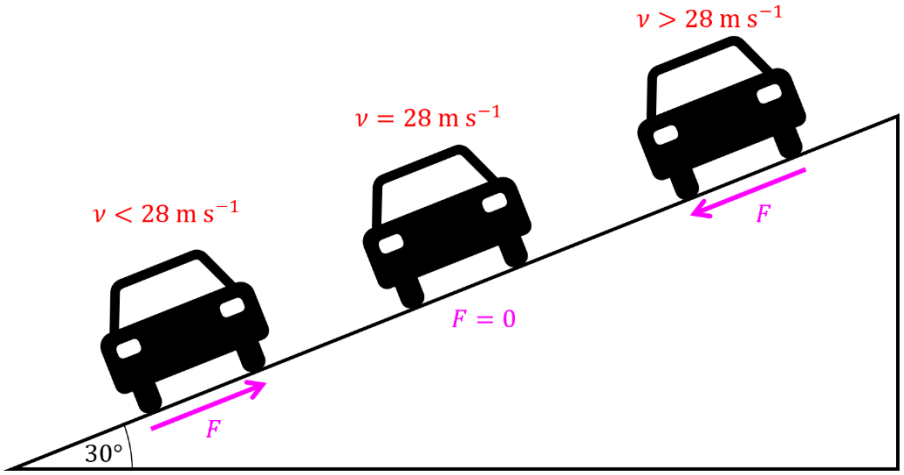
Then resolving horizontally (since the car is moving in a horizontal circle) and using $a = v^2/r$ gives,

$$R\sin\theta = \frac{mv^2}{r}$$

This should look familiar: it is exactly the same maths as the conical pendulum, but instead of tension *T*, it's the normal reaction, *R*. Turning the mathematical handle in the same way as with the conical pendulum and dividing the horizontal equation by the vertical gives us

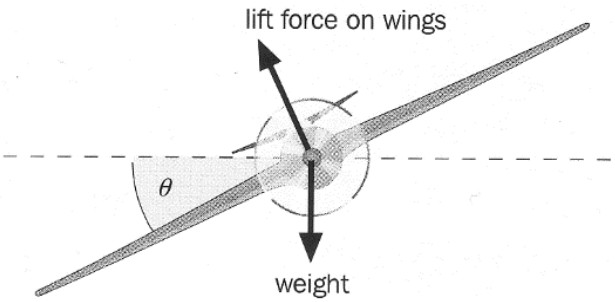
$$\tan\theta = \frac{v^2}{gr} = \frac{\omega^2 r}{g}$$

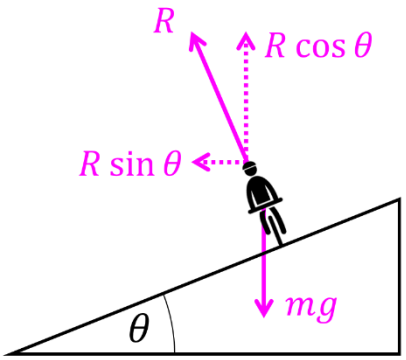
For example, for a corner banked at θ = 30° and with r = 5 m, the speed required is 28 m s^{-1}. At this speed, the car will follow the banked turn without relying on friction at all. Below this speed, and you will be relying on friction pushing you up the slope to stop you from slipping down the slope. Above this speed, you'll be relying on friction pushing down the slope to stop you sliding up the slope.

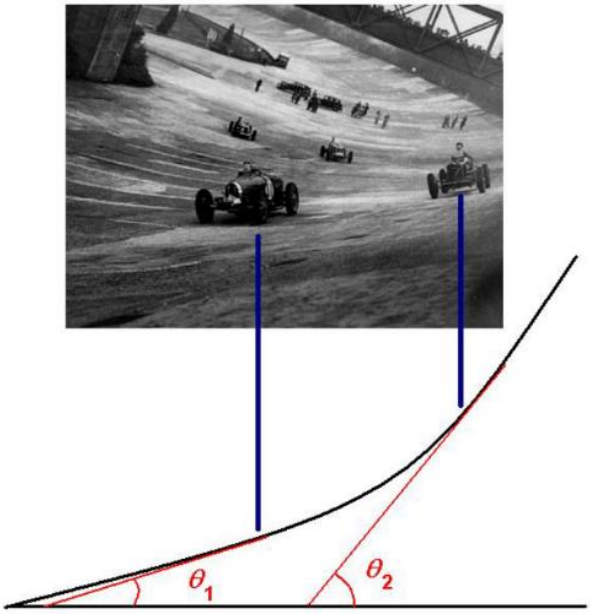


This also tells us that the faster you want to take a certain corner, the greater the angle θ needs to be, should you wish not to rely on friction. This is seen in old racetracks and velodromes where the angle of the banking increase as you move away from the centre of the turn.

But it is not just cars and bikes - trains can have banked turns to reduce wear on the wheels. Aeroplanes also need to take corners at an angle, so the horizontal component of the lift force provides some on the centripetal acceleration.







Example 6.3 – Eddy Merckx goes into a turn banked at 26° to the horizontal with a radius of 41 m at a speed of 14 m s^{-1}. Will he require friction to hold his line?

We will assume no friction and see if the numbers match. To do this, we proceed in the same way as before: draw a force diagram (above) and then resolve forces vertically and horizontally

$$\text{R}(\uparrow) \quad R\cos\theta = mg$$

$$\text{R}(\leftarrow) \quad R\sin\theta = \frac{mv^2}{r}$$

Then dividing gives us our standard result

$$\tan\theta = \frac{v^2}{gr}$$

Entering our values for v, g and r gives

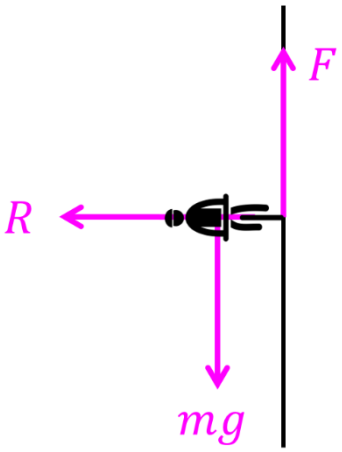
$$\theta = \tan^{-1}\left(\frac{14^2}{9.81 \times 41}\right) = 26°$$

which was our given angle for banking. Therefore, at this speed, Eddy will hold his line without relying on friction. Any faster and he will rely on friction pulling him down the slope to stop him sliding upwards.

Eddy Merckx is widely regarded as the greatest cyclist to have ever lived. In 1972, he set the hour record for cycling 49.4 km in 1 hour. This record stood for over 20 years despite massive technological evolution in that time. The current record is 55 km. What makes Merckx's record remarkable is he completed it at altitude in Mexico City on a standard issue track bike – no aerodynamic carbon fibre in sight. Furthermore, he completed the record after a full season of racing where he won the Tour de France and the Giro d'Italia, amongst countless other races. These days, folks attempting the record clear their calendar and train specifically for the event with a wealth of sports science behind them. His ferocious appetite for victory earned him the nickname of 'The Cannibal'.

Example 6.4 – In 2016 Guy Martin broke the recorded for the fastest speed on a wall of death. He travelled at 78 mph on a vertical wall in a circular track of diameter 37.5 m. The combined mass of Guy and his bike was 250 kg. Calculate:

a. the frictional force required;
b. the reaction force required;
c. the g-force Guy was subjected to.



If you are unfamiliar with The Wall of Death, I suggest a quick YouTube search. It's basically a banked turn where $\theta = 90$ °.

First up, let's see what we know:

$$v = 78 \text{ mph} = 34.869 \text{ m s}^{-1}$$

$$\text{diameter} = 37.5 \text{ m} \quad \Rightarrow \quad r = \frac{75}{4} \text{ m}$$

To answer this question, we will model the bike and rider as a particle. This means we can ignore the more complicated toppling effects that may arise. For the bike to remain in a horizontal circle, the weight pulling downwards must be equal to the frictional force pushing up. Therefore,

$$F = mg = 250 \times 9.81 = 2.5 \text{ kN}$$

This is quite a large force! Resolving forces horizontally we see that the centripetal force is provided by the reaction force. This is now in the horizontal direction as we are on a vertical road. Hence, we have

$$R = ma = \frac{mv^2}{r} = \frac{250 \times 34.869^2}{75/4} = 16 \text{ kN}$$

The g-force is calculated by working out the centripetal acceleration

$$a = \frac{v^2}{r} = \frac{34.869^2}{75/4} = 64.8 \text{ m s}^{-2} = 6.6g$$

A large but manageable amount a g-force. Most Wall of Death tracks, or 'drums' as they are called, are no more than 10 m in diameter. Attempting speed of 70 mph would spell certain death due to g-forces of $24g$. This is why Guy's Team built the massive Wall of Death track for him.

A proper treatment of The Wall of Death requires a deeper understanding of friction. This is added as an extra in Section 10.

Mastery Problems F – Banked Turns & Conical Pendula

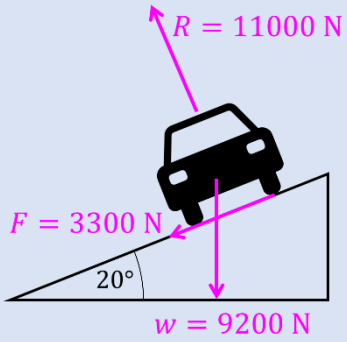
1. A particle of mass 0.20 kg is attached at one end of a light inextensible string of length 50 cm. The particle moves in a horizontal circle with an angular speed of 5.0 rad s^{-1} with the string inclined at θ to the vertical. Find the value of θ.

2. A particle of mass 3.0 kg is attached to one end of a light inextensible string of length 0.25 m. The particle moves in a horizontal circle and the string sweeps out the surface of a cone. The maximum tension that the string can sustain is 35 N. Find the maximum angular speed of the particle.

3. Calculate the time period of a conical pendulum which has a string length of 0.60 m and is at an angle of 58° to the vertical.

4. A certain conical pendulum has a time period T on Earth. Express the time period of the same conical pendulum on the Moon in terms of T.

5. The diagram below shows the forces acting on a car going round a banked turn. The radius of the turn is 10 m.

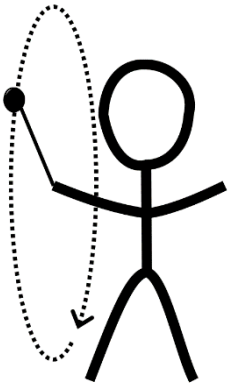


Calculate the resultant centripetal force and hence the speed of the car round the bend.

6. Felicity is about to attempt The Wall of Death in her hatchback and you've been asked to help her crunch the numbers. The circular track has a radius of 5.0 m and the combined mass of her and her car is 1200 kg. If Felicity travels at 15 m s^{-1}, calculate:

a. the frictional force required,
b. the reaction force required,
c. the g-force Felicity was subjected to.

8. Motion in a Vertical Circle



So far, we have just looked at objects moving in horizontal circles. Here, the weight always points perpendicularly to the plane of the circular path and therefore never contributes to the centripetal force. However, this is not the case for vertical circles – let's take a look.

8.1 Mass and String

Imagine we've got our mass on a piece of string but instead of whirling it around our head pretending to be a cowboy, let's whirl it round in a vertical circle in front of us.

The important difference with spinning the mass in a vertical circle is that it would be difficult for us to try and spin it with a constant **speed**. Gravity is pulling the mass down, so the mass will be moving faster at the bottom of the circle that at the bottom. The mass may be going round at a constant frequency of once per second, but the speed is changing throughout its journey.

Furthermore, you will notice when spinning the mass around in a vertical circle that the **tension** in the string does not remain constant either. As a general rule:

> ***For objects moving in a vertical, do not assume that the tension or the speed remain constant.***

This is important as it changes the way we approach objects moving in a vertical circle. Here is the general rule:

> ***For objects moving in a vertical circle, resolve forces and apply our circular motion equations at certain points.***

We will leave our vertical mass on a string here and move to the fairground – woop!

8.2 Loop the Loop

The best place to start with vertical circles is a loop the loop on a rollercoaster or a skatepark. Watch the following video. Why does a skateboarder fall down at the top of the loop the loop? If you watch, you can see it's because gravity pulls them down faster than the track curves round in front of them. In other words, they are not going fast enough so they **lose contact** with the track. Let's consider the forces acting on the boarder at the top, bottom and halfway either side of the loop.

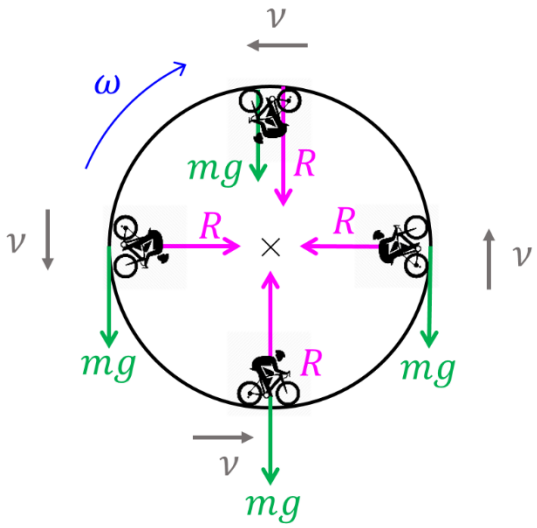


Figure 21 - Forces during a vertical circle. Note: neither the speed or the reaction force are constant.

As we can see in the diagram, the reaction force from the surface always contributes to the centripetal force. However, the weight always points downwards. Therefore, hallway up or down the loop, there is component of the weight acting horizontally. At the top of the loop, the weight contributes to the centripetal force whereas at the bottom, it opposes it.

For now, we will resolve forces at the top of the loop, as this is why the boarders fall off. This gives us

$$R_{\text{top}} + mg = \frac{mv_{\text{top}}^2}{\text{r}}$$

Note that in this expression, I have denoted the reaction force and speed at the top of the loop because they vary throughout the loop.

What does this expression tell us? Well, we said that there is a minimum speed required to make a given loop the loop. Below this speed, you lose contact with the track. Hence, to find this minimum speed, we can set $R_{\text{top}} = 0$, which gives us

$$mg = \frac{mv_{\text{top}}^2}{\text{r}}$$

Everybody brought mass to the party, leaving us with

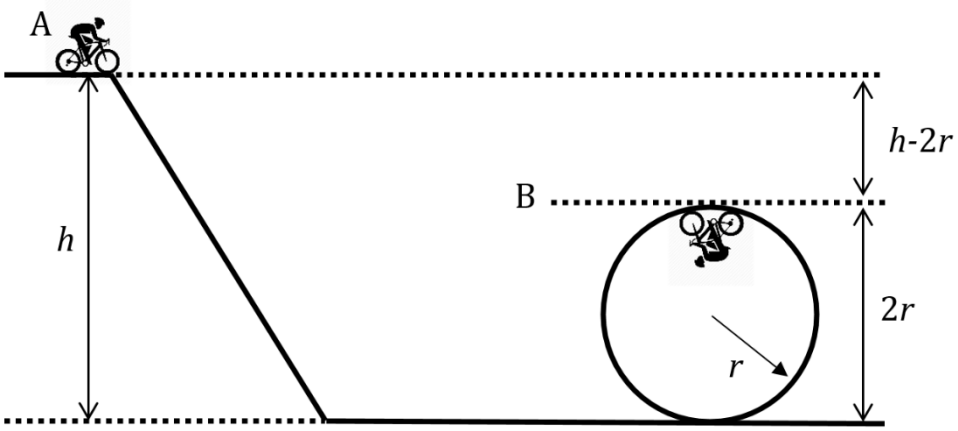
$$g = \frac{v_{\text{top}}^2}{\text{r}}$$

Before we rearrange for v_{top}, have you noticed anything? $\frac{v_{\text{top}}^2}{\text{r}}$ is the centripetal acceleration of the bucket so the above statement says that when $R_{\text{top}} = 0$, the centripetal acceleration is equal to g. This links to what we said before about the boarder falling faster than the track curves. This expression further demonstrates this as if the centripetal acceleration is less than the acceleration due to gravity, the skater will fall downwards. Therefore $R_{\text{top}} = 0$ and $a_{\text{top}} = g$ are equivalent criteria for the boarder successfully making a loop the loop.

Getting back on track, let's rearrange for v_{top} to get

$$v_{\text{top}} = \sqrt{gr}$$

which is he minimum speed at the top of the loop in order to make the loop. This isn't very useful for us – we want to know how high the ramp needs to be to reach this speed at the top. To solve this problem, we are going to need to consider the energy changes from the top of the ramp to the top of the loop. We will assume that there is not energy lost to friction.



By the conservation of energy, we know that the energy at point A (top of the ramp) must be equal to the energy at point B (top of the loop). If we assume the rider starts from rest at A, then we know the kinetic energy at B is equal to the change in gravitational potential energy between A and B. In other words:

$$E_{\text{A}} = E_{\text{B}}$$

$$mgh = \frac{1}{2}mv_{\text{B}}^2 + mg(2r)$$

$$\Rightarrow v_B^2 = 2g(h-2r)$$

$$\Rightarrow v_B = \sqrt{2g(h-2r)}$$

Now by substituting the above equation into $v_{top} = \sqrt{gr}$, we get

$$\sqrt{2g(h-2r)} = \sqrt{gr}$$

$$2(h-2r) = r$$

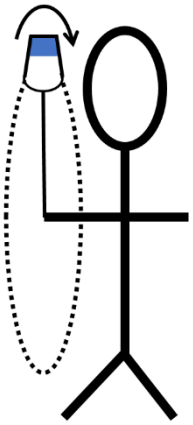
Giving us our final answer for the height of the ramp to be

$$h = \frac{5}{2}r$$

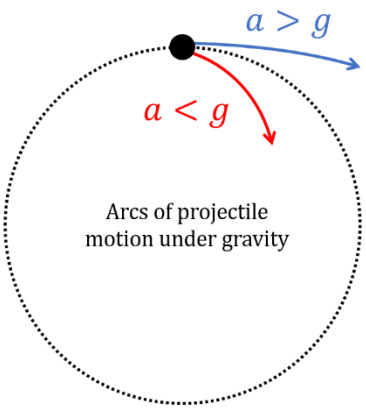
An interesting result – the ramp only needs to be half a radius higher than the top of the loop in order to successfully for the loop. In reality, it may need to be slightly higher to account for frictional effects. We could have taken an extra step and calculated the kinetic energy at the bottom of the loop first, to work out the entry speed of the rider.

8.3 Bucket of Water

A personal favourite party trick of mine is spinning a bucket full of water over my head in a vertical circle. Excellent fun. But why does the water stay in the bucket?



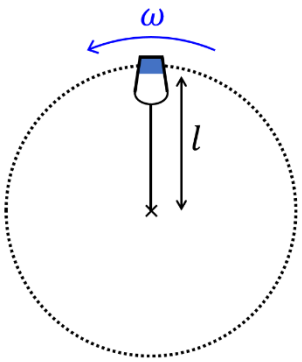
One way to consider the problem is similar to how we looked at the skater problem. The water stays in the bucket because the centripetal acceleration is greater than the acceleration due to gravity. The bucket takes the water round in a tighter arc than gravity alone would take the water. If the bucket was swinging round with $a < g$, then the arc of projectile motion would be smaller than the arc of the circular path and the water would fall out.



This video explains this point well:

https://www.youtube.com/watch?v=Zulw5bQ18Kk

Now, let's calculate the minimum angular speed required to keep the water in the bucket. Let's say the bucket is on a length of string l and the mass of water is m. We



The reaction force from the bucket on the water always acts inwards.

why at the top

$$R_{top} + mg = m\omega_{top}^2 l$$

$$mg = m\omega_{top}^2 l$$

which cancels to

$$g = \omega_{top}^2 l$$

$$\omega_{\text{top}} = \sqrt{\frac{g}{l}}$$

This gives us the minimum angular speed at the top of the circle for water not to fall out of the bucket

It is important to stress that the necessary criteria for the water not to fall out the bucket is the reaction force on the water from the bucket $R_{\text{top}} \leq 0$ OR $a_{\text{top}} \geq g$. We showed in the skater problem that these two criteria are equivalent!

https://www.youtube.com/watch?v=TIBcntHCxjQ

This video states that the condition for water falling out is when the tension in the rope is zero. Technically speaking, the tension force is acting on the bucket and not the water in the bucket. It gives us the same result though.

Mastery Problems G – Vertical Circles

1. A bucket of water is swung in a vertical circle or radius r in such a sway that the bucket is upside down when it is at the top of the circle. Derive an expression for the minimum speed of the bucket at this point if the water is to remain inside the bucket. *See how far you can get without looking back at the notes!*
2. An aeroplane loops the loop in a vertical circle of radius 200 m, with a speed of 40 m s^{-1} at the top of the loop. The pilot has a mass of 80 kg. Calculate the tension in his strap holding him into his seat at the top of the loop.
3. A car just leaves the road when it passes over a semi-circular hump of radius 36 m. Calculate the speed of the car.
4. The TV show 'Fifth Gear' decided it would be an excellent idea to get someone to drive a family car over a loop the loop track 43 feet high.
   a. Calculate the speed the car needs to be going at the bottom of the loop to maintain contact at the top.
   b. Calculate the g-force experienced by the driver as he enters the loop.
   c. Watch the video to see if you are correct! https://www.youtube.com/watch?v=wiZoVAZGgsw

9. Mixed Problems

First, complete the end of topic *Examination Style Questions* in Breithaupt, pages 30-33.

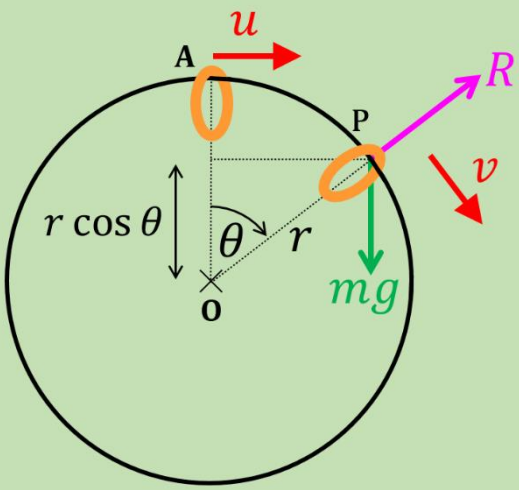
The Mixed Problems that follow are intended to be a challenge – seek assistance from your peers or myself should you have any problems.

Example 9.1 – A small bead of mass m is threaded on a smooth circular wire of radius r and centre O, and is fixed in the vertical plane. The bead is projected with a speed u from the highest point, A, of the wire. Find the reaction on the bead due to the wire when the bead is at P, in terms of m, g, r, u, and θ where $\theta = \text{A}\hat{\text{O}}\text{P}$.

A bead on a wire is new territory for us so let's try and get a good physical picture of what's going on. I find it helps to imagine those wiggly counting machines you may have seen as a child – that's what the question means by a bead on a wire.

The interesting difference with a bead on a wire is that the bead will always stay on the wire. In previous vertical circle problems, if the speed wasn't sufficient, the boarder fell off the loop or the water fell out the bucket. This difference arises because the reaction force on the bead from the wire can push the bead inwards or outwards. If the bead is going too fast, the reaction force pulls it inwards. Too slow and the reaction force pushes outwards.

Let's draw a diagram of the situation.



Here we have let the speed at P = v and the reaction = R. We shall see that the decision to have the reaction pointing radially outwards rather than inwards doesn't matter.

As ever with vertical circles, we begin with energy conservation.

$$\text{Change in KE A} \to \text{P} = \text{Change in GPE A} \to \text{P}$$

$$\text{KE at P} - \text{KE at A} = \text{Change in GPE}$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mg(r - r\cos\theta)$$

$$v^2 = u^2 + 2gr(1 - \cos\theta)$$

The we apply Newton's 2nd Law at the point P, giving

$$F = ma$$

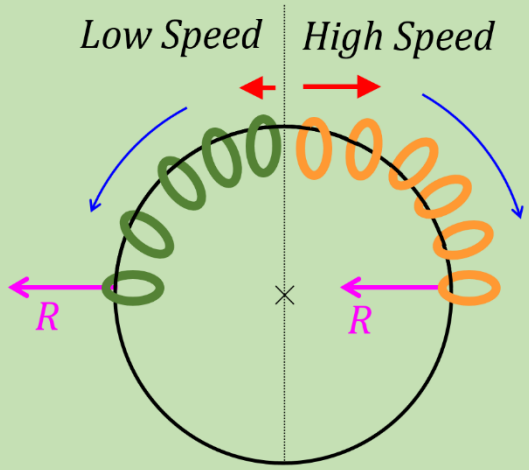
$$mg\cos\theta - R = \frac{mv^2}{r}$$

Inserting our equation for v^2 from energy conservation gives

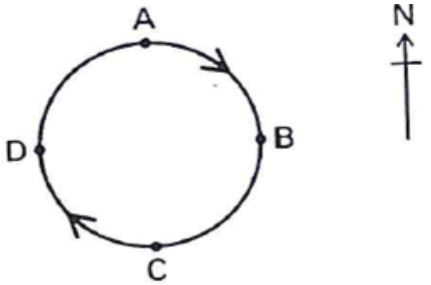
$$mg\cos\theta - R = \frac{m}{r}[u^2 + 2gr(1 - \cos\theta)]$$

$$R = mg(3\cos\theta - 2) - \frac{mu^2}{r}$$

We see from the final answer that the reaction force can be either positive or negative. This means it can either act inwards or outward, depending on the speed of the bead. If u increases so the second term is greater than the first, then the reaction force will be negative and hence be directed inwards. This makes physical sense as a larder speed requires ore centripetal force and therefore the reaction is required to contribute to this force. This is shown in the diagram below.



1. The diagram shows a particle moving in a circle ABCDA of radius 8.0 m at a constant



speed. It completes on revolution in 5.0 s. What is:

- **a.** its average speed for one revolution;
- **b.** its average speed from A to B;
- **c.** its average velocity for one revolution;
- **d.** its average velocity from A to C;
- **e.** its average velocity from A to B;
- **f.** its change in velocity from A to C;
- **g.** its change in velocity from A to B?

2. Calculate the force necessary to keep a mass of 0.80 kg revolving in a horizontal circle of radius 0.70 m with a period 0.50 s. State the direction of the force.

3. Use Newton's Laws to explain why a body moving with constant speed along a circular path must experience a force directed towards the centre of the circle.

4. A simple pendulum, suspended from a fixed point, consists of a light cord of length 500 mm and a bob of weight 2.0 N. The bob is made to move in a horizontal circular path. If the maximum tension which the cord can withstand is 5.0 N, show whether or not it is possible for the radius of the path of the bob to be 300 mm.

5. A particle of mass 0.30 kg moves with an angular velocity of 10 rad s^{-1} in a horizontal circle of radius 20 cm inside a smooth hemispherical bowl. Find the reaction of the bowl on the particle and the radius of the bowl.

6. A particle is attached by means of a light, inextensible string to a point 0.40 m above a smooth, horizontal table. The particle moves on the table in a circle of radius 0.30m with angular speed ω. Find the reaction on the particle in terms of ω. Hence, find the maximum angular speed for which the particle can remain on the table.

7. A cork suspended in water can be used as a primitive accelerometer. Watch the video and explain how this works: LINK

8. An aeroplane has a mass m and is moving at a constant speed v in a horizontal circular path of radius r. It does so by banking at an angle θ to the horizontal. Show that $v = \sqrt{rg \tan\theta}$.

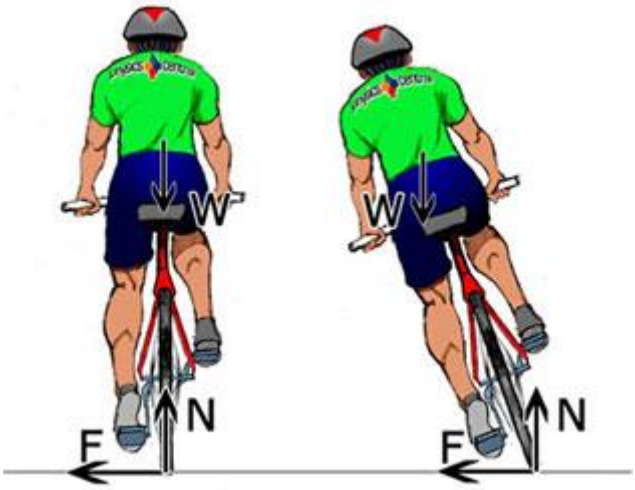
9. Being weightless on space station sure looks fun, but it can have adverse health effects over longer periods of time. To combat this, artificial gravity space stations have been proposed that utilise a rotating disc to create a g-force. The astronauts live at the outer edge of the station where the centripetal force provided by the reaction force gives them the impression of weight. If the diameter of the disc is 20 m, calculate the necessary angular speed in order to achieve artificial gravity equal to that on the Earth's surface.

10. Colin gets bored of swinging masses above his head for the sake of a Physics problem so decides to be different and swing a mass on the end of a spring around his head in a horizontal circle. The spring has a spring constant of 1.6 N m^{-1} and has a natural length of 1.0 m. Calculate the radius of the circle if the mass is 200 g and moves at a constant speed of 4.0 m s^{-1}.

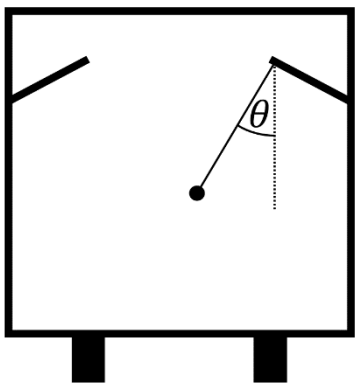
11. A compressed spring is used to propel a ball-bearing long a track that contains a circular loop of 0.10 m in a vertical plane. The spring obeys Hooke's Law and requires a force of 0.20 N to compress it 1.0 mm.

- **a.** The spring is compressed by 30 mm. Calculate the energy stored in the spring.
- **b.** A ball-bearing of mass 0.025 kg is placed against the end of the spring which is then released. Calculate
  - **i.** the speed at which the ball-bearing leaves the spring,
  - **ii.** the speed of the ball at the top of the loop.
  - **iii.** the force exerted on the ball by the track at the top of the loop.

12. In order to take a corner on a flat track at speed, a cyclist needs to lean into the corner. Not leaning into a corner can mean the cyclist gets thrown outwards off their bike. Use the diagram below of two cyclists taking a left turn to explain why this happens.



13. Bob and Fred are bored on a long train ride up to John O'Groats. Bob suggests they use an apple on the end of a piece of string hung from the luggage rack to measure the centripetal acceleration of the train. Bob says that the apple will hag vertically downwards when the train is travelling at a constant speed. However, when the train is going round a corner, the pendulum will tilt at an angle θ. The larger the acceleration, the larger the angle.



Fred measures the speed of the train on his phone using GPS to be constant at 90 mph. If the angle of the pendulum is 10°, calculate the radius of the turn.

14. A small bead is threaded on a smooth circular wire of radius r which is fixed in a vertical plane. The bead is projected from the lowest point of the wire with a speed of $\sqrt{6gr}$. Find the speed of the bead when it has turned through:

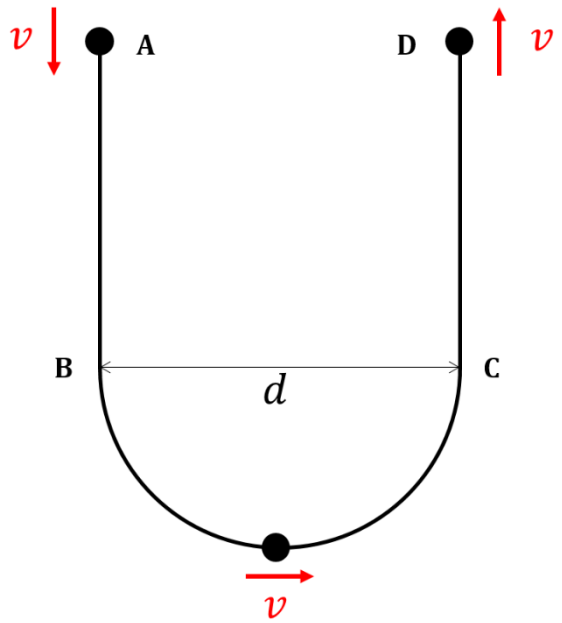
a. 60°
b. 90°
c. 180°
d. 300°

Hint: show that the velocity at a certain angle is given by $v(\theta) = \sqrt{2gr[\cos\theta + 2]}$

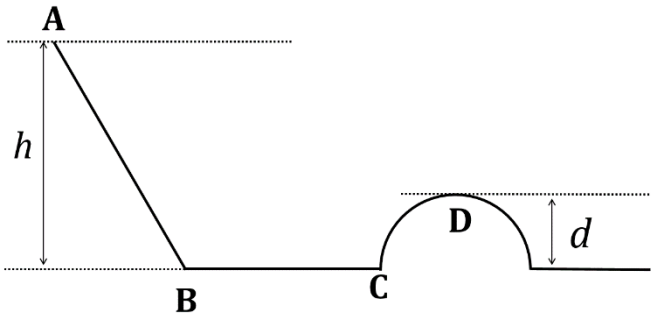
15. A particle of mass m travels at a constant speed v in a vacuum from point A to point B, as shown in the diagram opposite.

For the section of the path from B to C, calculate:

a. the time taken,
b. the change in momentum of the particle,
c. the magnitude of the force acting on the particle,
d. the work done on the particle by this force.



16. Sanka is attempting to win his first Jamaican National Push-Cart Championship. They start at the top of a hill and the first to the bottom wins – no engines of any kind are allowed so gravity is the name of the game. This year, the course includes a semi-circular bridge on the flat section of the course, as modelled in the diagram below.



The height of the start hill is $h = 110$ m and the height of the bridge is $d = 9.0$ m. Sanka and his cart have a combined mass of 110 kg and its maximum braking force is 500 N.

a. Calculate the speed of the cart at B.

b. Calculate the maximum speed of the cart at the top of the bridge for the cart to maintain contact with the road.

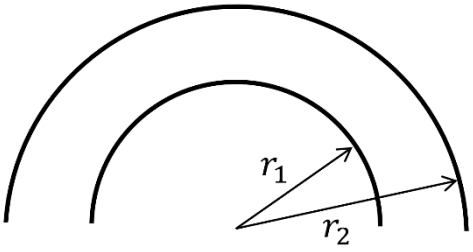
c. In a bizarre turn of events, any cart which loses contact with the ground at D will be disqualified. Calculate the distance before C Sanka will need to start braking with the maximum braking force in order to start the bridge at C with the correct speed to maintain contact at D.

17. An orange of mass m is moving on the inner surface of a perfectly smooth, hemispherical fruit bowl with a radius $2a$. The orange is moving in a horizontal circle with angular speed $\sqrt{g/a}$.

a. Calculate the magnitude of the force on the orange by the surface of the bowl.

b. Calculate the height of the centre of mass of the orange above the surface of the table which the bowl rests on.

18. A turn on a race track is banked at 60° to the horizontal and its surface is made of solid ice. The inside of the bend has a radius of $r_1 = 34$ m and the outside of the bend has a radius of $r_2 = 48$ m. The turn is semi-circular when viewed from above. Assume there is no friction between the tyres and the road due to the ice.



a. Show that the speed of a car required not to slip at a given radius r is given by the expression

$$v(r) = \sqrt{gr \tan\theta}$$

b. Derive an expression for the time taken for a car to go round the semi-circular turn at radius r.

c. Calculate the shortest possible time to navigate the turn, assuming the driver holds the same line (i.e. their radius remains constant) throughout the turn.

19. A particle of mass m is attached at one end of a piece of string of length l. The other end of the string is attached to a fixed wooden peg. The particle is hanging in equilibrium before it is set in motion with a horizontal speed u. It then moves in a vertical circle of radius l. Show that the tension in the string T varies with angle from the vertical, θ by the following equation:

$$T(\theta) = 3mg\cos\theta + \frac{mu^2}{l} - 2mg$$

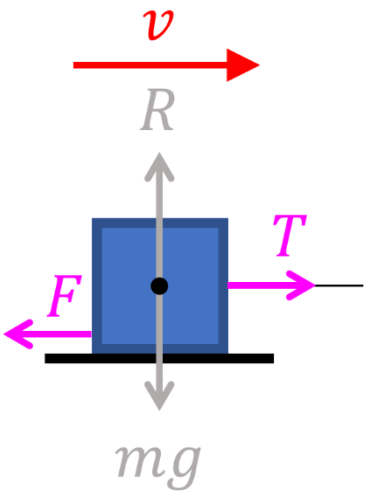
Hint: resolve forces and look at the energy changes between the equilibrium position with speed u and another point an angular displacement of θ away.

The following sections cover material beyond the scope of the A-Level Physics course. They are designed to deepen the understanding of circular motion for A/A* physicists and give a flavour of university physics/engineering.

10. *Friction

One thing we don't look into much in A-Level Physics is friction. It's a great shame as it adds another dimension to problem solving – particularly in the context of circular motion. In this section, we will look at a simple quantitative treatment of friction before looking at applying it to circular motion problems. We will keep our treatment of friction here in line with what you may study in Mechanics for A-Level Maths.

10.1 Investigating Friction

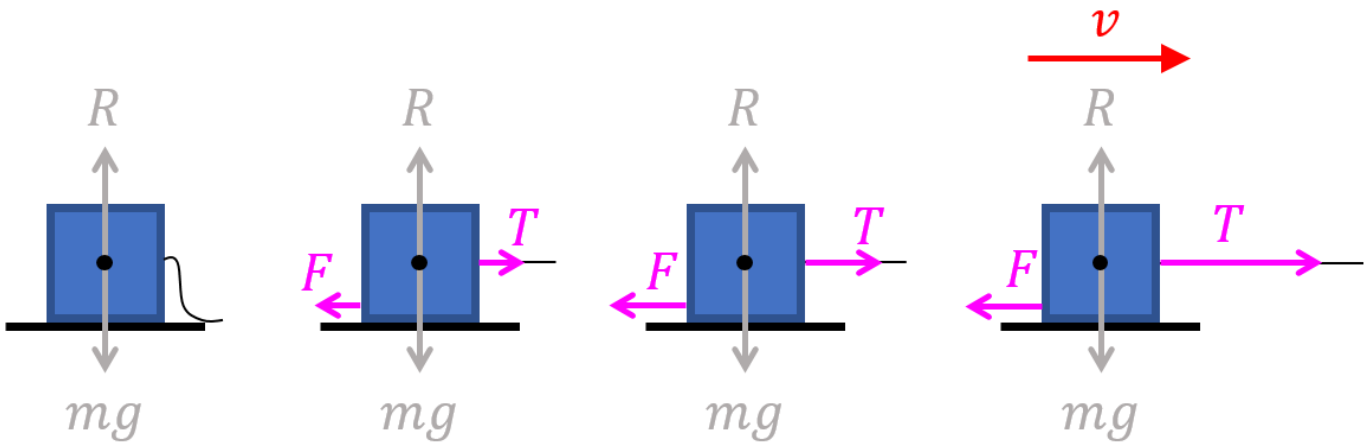


When the surface of a body moves or tends to move over another surface, each body experiences a frictional force. The frictional forces act along the common surface and act in the opposite direction to the relative motion.

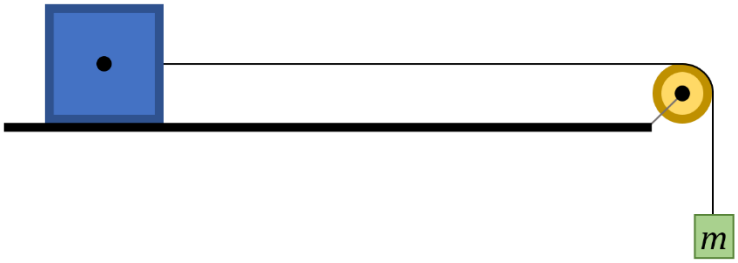
The diagram opposite shows the forces acting on a block that is being pulled along a rough table. The tension force T pulls the block to the right whilst the friction F exerts a force on the base of the block to the left.

To investigate the properties of friction, find an object roughly a kilogram in mass and place it on the table – a bag of sugar is a good place the start. If you push the mass with a very small force, the mass remains stationary. This is because you're pushing the mass forwards, but friction is pushing mass backwards by the same amount.

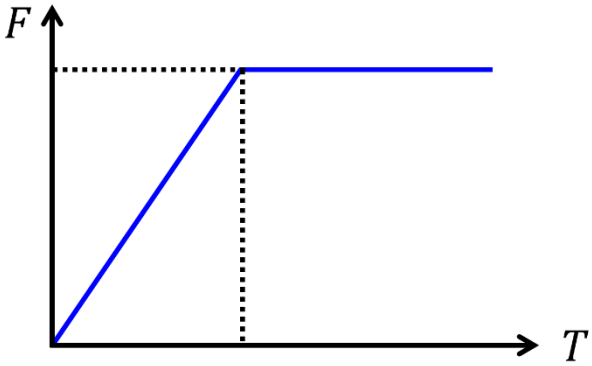
If you push with a slightly larger force, the mass remains stationary again, which means friction pushes back with a larger force. Friction will increase up to a certain **maximum** value – if you slowly increase the force on the mass, it will move at certain value of push force. These observations are shown in the diagram below, with a block being pulled by a string with increasing tension force T as you move from left to right. The block finally starts moving in the diagram on the far right. Here, the tension force is larger than friction (which has reached its maximum value) so the block accelerates to the right.



We could perform more accurate investigation of what we have described using the setup below. The masses hanging off the table allow a known constant force to be applied to the block.



The results we would obtain are plotted below. The frictional force increases to a maximum value where after it remains constant.



Using this setup, we could also try and investigate what factors effect this maximum value of the frictional force between two surfaces. We could change the area of contact by using different block faces and rest masses on the block to change the reaction force.

The results of the experiment can be summarised into three experimental laws of friction:

1. The frictional force between two surfaces always opposes the relative motion or attempted motion.
2. Frictional forces are independent of the area of contact of the two surfaces.
3. For two surfaces which are not moving, the maximum frictional force is directly proportional to the normal reaction.

   For two surfaces which are moving, the sliding frictional force is directly proportional to the normal reaction and is approximately independent of the speed of the object.

Note: An explanation of these frictional effects is beyond the scope of the A-Level Mathematics specification and is therefore is not discussed here. However, some of the laws of friction may have come as a surprise (they did to me!) and I therefore suggest some googling if you are interested. ***What actually causes friction?***

10.2 Modelling Friction

From above laws, we have that frictional force F is directly proportional to the reaction force, R. We call the constant of proportionality the coefficient of friction, μ. It is a constant for two given surfaces but does depend on conditions such as the presence of moisture and lubricants. Our model for friction states then states that

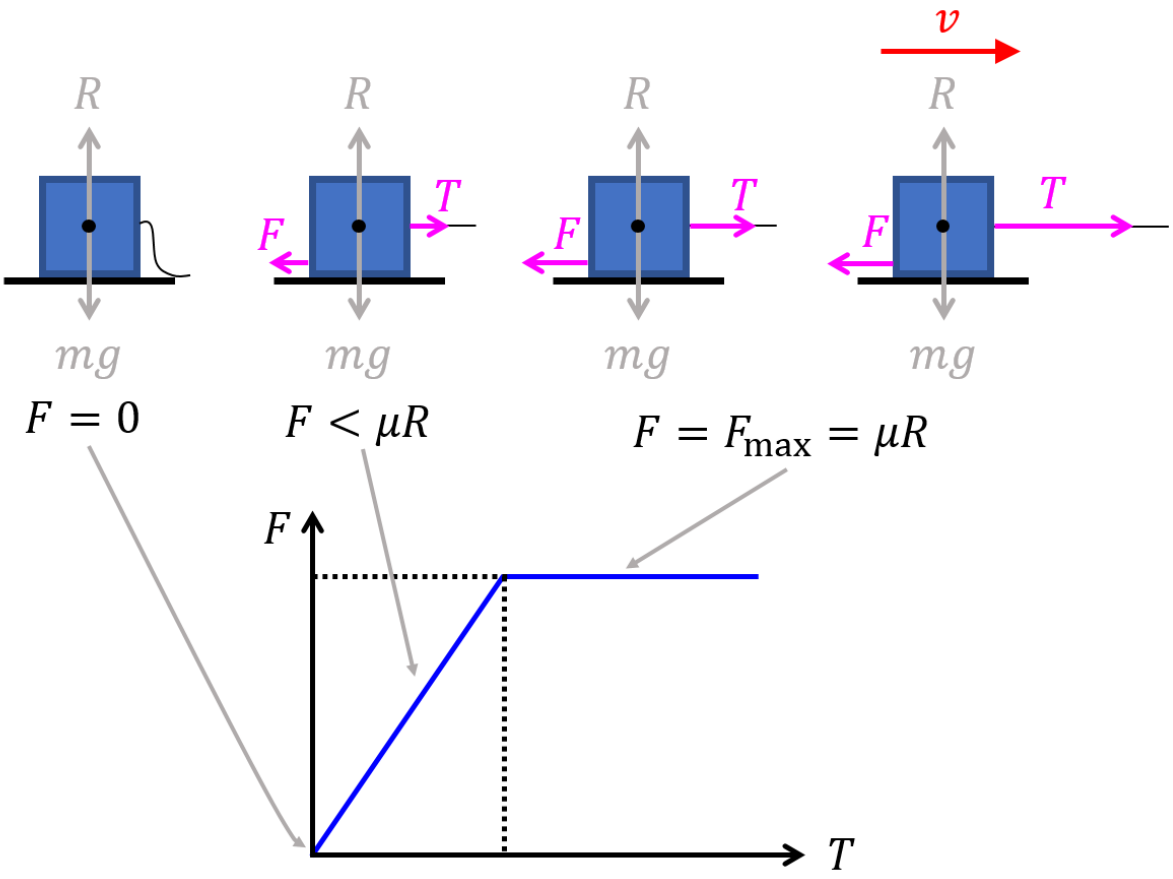
$$F \leq \mu R$$

The limiting or maximum value of friction is given by

$$F = F_{\text{MAX}} = \mu R$$

This is the most useful relation as most problems we encounter involve an object that is moving or is about to move. In both of these cases, the frictional force is at its maximum value.

The diagram below links the model we now have back to our block example from the previous section. It should be noted that the diagram third from the left shows a block in 'limiting equilibrium'. This is the point at which the block is just about to move and therefore friction is at its maximum value and still equal and opposite to the force which it opposes.



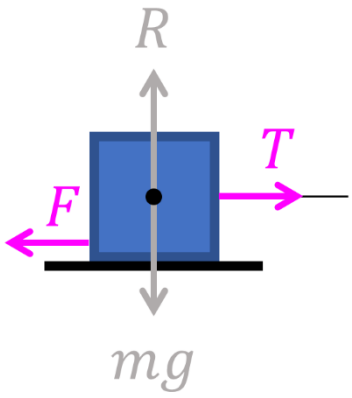
Key Points from Friction Model

1. If an object is at rest, the frictional force is equal and opposite to the force which it opposes parallel to the surface.
2. If an object is in limiting equilibrium, the object is just about to move and therefore $F = F_{\text{MAX}} = \mu R$ and is also equal and opposite to the force which it opposes.
3. If an object is moving, $F = F_{\text{MAX}} = \mu R$

10.3 Friction Problems

Before we move onto to looking at circular motion and friction combine, we will first look at a few simpler problems involving our new model of friction.

Example 10.1 – A block of mass 1.5 kg rests on a rough horizontal table. A horizontal force of 12 N is applied to the block puts the block in limiting equilibrium. Find the coefficient of friction between the block and the table.



Standard operating procedure – draw a force diagram. Next, we resolve force vertically:

$$\text{R}(\uparrow) \qquad R = mg$$

We then use the fact the block is in limiting equilibrium. This implies that $F = F_{\text{MAX}} = \mu R$. Resolving horizontally and using this gives

$$\text{R}(\rightarrow) \qquad T = F$$

$$T = \mu R$$

Then we pop $R = mg$ into the above equation to yield

$$T = \mu mg$$

$$\Rightarrow \mu = \frac{T}{mg} = \frac{12}{1.5 \times 9.81} = 0.86$$

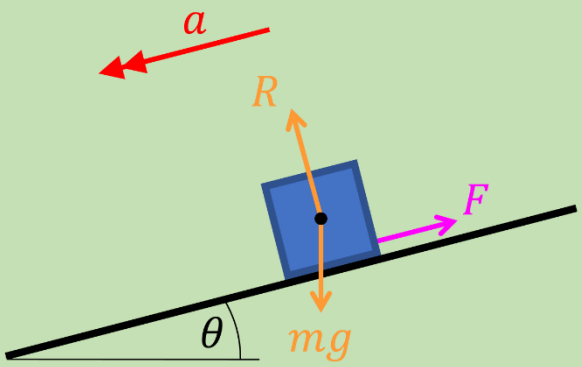
Note that μ has no units as the is the ratio of two forces.

This simple problem gives us an insight to why friction adds an extra dimension to problem solving. Using the fact that friction depends on the reaction force links the horizontal and vertical forces. Beautiful stuff!

Second only to ladder problems are rough inclined plane problems. The excitement is unbearable. Let's get stuck in.

Example 10.2 – A block of mass m sits on a rough plane inclined at an angle of 45° to the horizontal. If the coefficient of friction between the two surfaces is 0.66. Calculate the acceleration of the block.

Again, standard operating procedure – draw a force diagram.



As with all inclined plane problems, we need to resolve forces parallel and perpendicular to the slope. Resolving perpendicularly to the slope, we have no acceleration, therefore

$$\text{R}(\perp) \qquad R = mg\cos\theta$$

Resolving down the slope, we have an acceleration of a, therefore

$$\text{R}(\parallel) \qquad mg\sin\theta - F = ma$$

To proceed, we need to use our new knowledge of friction. If the block is moving, friction is at its maximum and hence

$$F = F_{\text{MAX}} = \mu R = \mu mg\cos\theta$$

Plugging this into our second equation gives

$$mg\ \sin\theta - \mu mg\cos\theta = ma$$

Everybody brought mass to the party! Accounting for this an rearranging for a leaves us with

$$a = g(\sin\theta - \mu\cos\theta)$$

Substituting $\theta = 45°$, $g = 9.81\ \text{m s}^{-2}$ and $\mu = 0.66$ gives us our answer of

$$a = 9.81(\sin 45° - 0.66 \times \cos 45°) = 2.4\ \text{m s}^{-2}$$

Extension Problems I - Friction

1. A mass of 10 kg is pushed at a constant speed along a rough table by a horizontal force of 73 N. Calculate the coefficient of friction between the block and the table.

2. A wooden block slides across a table until friction brings it to rest. The coefficient of friction between the block and the table is 0.55. Calculate the deceleration of the block as it was sliding.

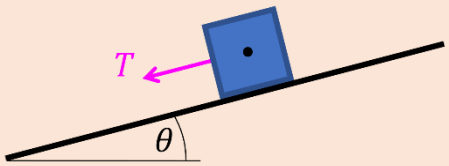
3. Colin pulls a sled full of supplies across the snow in Alaska using a rope at an angle θ to the horizontal. The total mass of the sled and supplies is 145 kg. The coefficient of friction between the sled and the snow is 0.12 and Colin's maximum pull force is 800 N. Calculate the maximum possible value of θ for the sled to continue at constant speed under the action of Colin's maximum pull force.

4. Calculate the acceleration of a wooden block resting on a rough plane with $\mu = 0.25$ that is inclined at an angle of 30° to the horizontal.

5. Consider a large ice cube being towed across a table by a hanging mass and pulley system shown below.

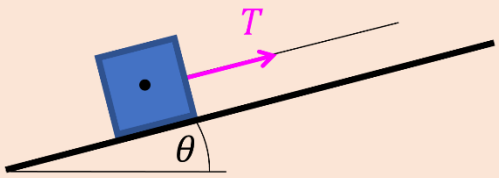
The large blue block has a mass $M = 5.0$ kg and the coefficient between the ice and the table is 0.12. Calculate the amount of hanging mass m required to just make the ice block move.

6. A 3.0 kg wooden block is towed down a slope by a force T = 16 N at a constant speed of 0.50 m s^{-1}.



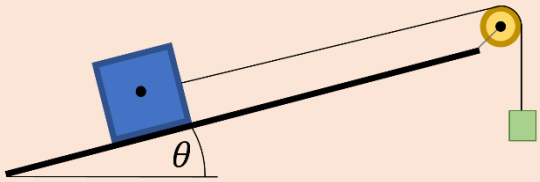
If the plane is inclined at an angle $\theta = 15°$, calculate the coefficient of friction between the block and the slope.

7. A 5.0 kg wooden block is towed up an inclined plane by a piece of string parallel to the slops, as shown in the diagram below.



If $\mu = 0.44$ and $\theta = 24°$, calculate the tension force required to accelerate the block at 0.50 m s^{-2}.

8. Consider a wooden block being towed up an inclined plane by the hanging mass/pulley system shown below.



The mass of the blue block is $M = 10$ kg and the mass of the green hanging block is $m = 3.5$ kg. $\theta = 30°$ and the coefficient of friction is $\mu = 0.33$. By assuming the magnitude of the two block's accelerations are the same, calculate the acceleration of the blue block.

9. The coefficient of friction between a car tyre and a dry road is approximately 0.7. Use this to estimate the stopping distance of a car travelling at 50 mph.

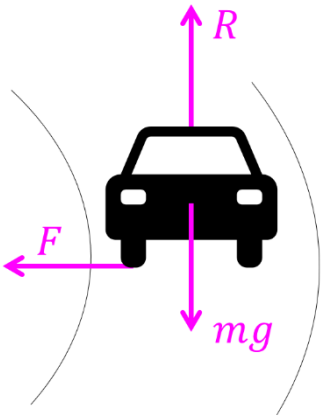
Hint: assume that the car is braking so hard that it is skidding.

Here's a video form the excellent Walter Lewin Lecture series where he shows off some good friction demonstrations.

https://www.youtube.com/watch?v=FWh-enOdXM4

We can now revisit some of the circular motion problems with our new model for friction.

10.4 Bends Reloaded



Let us first revisit a car travelling in a horizontal circle of radius r around a bend on a flat road at a speed v. Resolving forces vertically yields

$$\text{R}(\uparrow) \qquad R = mg$$

which will come in handy later. We then resolve horizontally and use the equation for centripetal acceleration, giving

$$\text{R}(\leftarrow) \qquad F = \frac{mv^2}{r}$$

So far, the same as previously. Let's find the maximum speed a driver can take a given corner. At this speed v_{max}, the car is about to slip outwards so friction must be at its maximum, meaning $F = F_{\text{MAX}} = \mu R$. Using our vertical equation then gives us

$$F = \mu R = \mu mg$$

Substituting this back into the horizontal equation gives

$$\mu mg = \frac{mv_{\text{max}}^2}{r}$$

and a little rearranging leaves us at

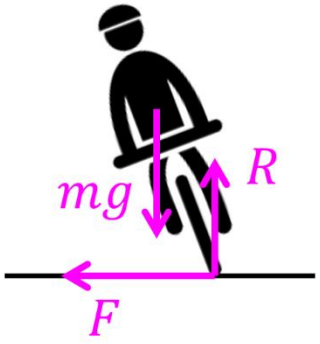
$$v_{\text{max}} = \sqrt{\mu g r}$$

So, we have shown that the maximum speed one can take a given corner depends only on the coefficient of friction between the tyres and the ground and g. In good conditions, the coefficient of friction between a tyre and the road is 0.7-0.8. However, this can decrease to around 0.4 and lower in wet conditions. This demonstrates the need for caution when cornering in wet conditions.

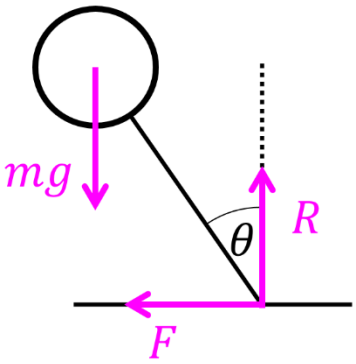
The dependence of g may initially strike you as odd, but it stems from the dependence of friction on the reaction force. If g was smaller, friction would be less as the reaction force would be less. Coupled with the lower coefficient of friction of the Moon's surface soil, this suggests the Moon would not be the ideal place to start a Space F1 race.

We can do the same and apply our new model of friction to banked turns. I think I will leave this one to you in the next problems section.

10.5 Motorbike Turns



Problem 12 in the Mixed Problems looks at why cyclists lean into corners. It should have been apparent from the diagram that when the rider does not lean, friction provides and overall torque that would cause the rider to rotate and topple over. Leaning into the corner means the reaction force on the wheel can counteract this rotation by providing a torque in the opposite direction. But why does a motorbike lean more than a bicycle? A quick search on YouTube will show you that a motorcyclists knees are often very close to the road when cornering.



To answer this, we will model the motorcyclist and bike as a ball on the end of a light rod, as shown in the second diagram. This means that the weight acts from the centre of the ball which is a distance l away from where the wheel contacts the ground. The clockwise moment provided by the reaction force should balance the anti-clockwise moment provided by the friction. Taking moments about the centre of mass gives

$$l \times R \sin\theta = l \times F \sin(90° - \theta)$$

Then $\sin(90° - \theta) = \cos\theta$ and l cancels, leaving

$$R \sin\theta = F \cos\theta$$

Resolving forces vertically we have no acceleration, therefore

$$R(\uparrow) \qquad R = mg$$

which we can substitute back into the above equation to give

$$mg \sin\theta = F \cos\theta$$

Friction provides the centripetal force for the biker to continue moving in a circle and therefore

$$\mathrm{R}(\rightarrow) \qquad F = \frac{mv^2}{r}$$

Substituting this back in gives us

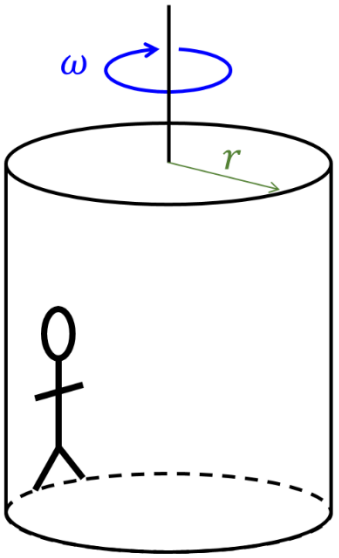
$$mg \sin\theta = \frac{mv^2}{r} \cos\theta$$

Everybody brought mass to the party! Accounting for this and rearranging using $\tan\theta = \sin\theta \,/\, \cos\theta$ leaves us with

$$\theta = \tan^{-1}\left(\frac{v^2}{gr}\right)$$

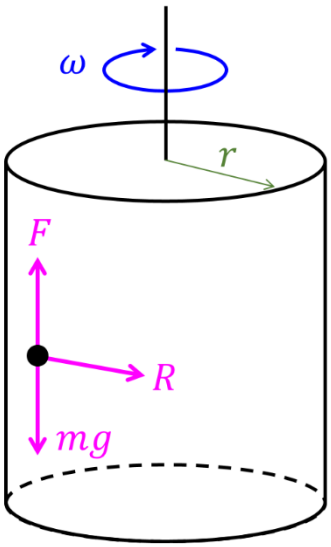
In this expression, as v increases, so does θ. We can therefore see that the faster a rider takes a corner, the further down they need to lean to counteract the increasing frictional force. This explains why a motorbike needs to lean further into a corner than a cyclist.

10.6 The Spinning Terror



Consider a fairground ground called the 'Spinning Terror'. It consists a hollow cylinder that rotates at an angular speed ω. As the speed increases, the reaction force on the person also increases, as this provides the centripetal force. All good so far.

The ride starts with a floor at the bottom of the cylinder. This provides a reaction force to stop the person falling downwards. However, at some value of ω, this floor is removed. Their weight is pulling them downwards but the friction between them and the wall opposes this tendency to slip downwards. Since the friction increase with reaction force and reaction force in turn increase with ω, there is some minimum value of ω where it is safe to remove the floor. Cool stuff! Let's find this minimum value of ω.



The second diagram shows the forces acting on the person. Resolving vertically gives us

$$\text{R}(\uparrow) \qquad F = mg$$

The resolving into the centre of the circle gives

$$\text{R}(\rightarrow) \qquad R = m\omega^2 r$$

As with all our other questions on friction, the link between these two equations comes from our model for friction. At the minimum angular speed, the person is just about to slip downwards, therefore friction is at its limiting value and therefore

$$F = F_{\text{max}} = \mu R$$

Popping this in the first equation gives us

$$\mu R = mg$$

And then inserting the second equation for R in gives

$$\mu m\omega^2 r = mg$$

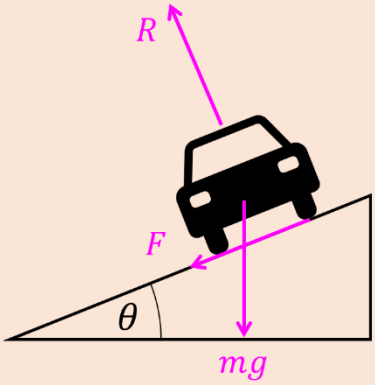
Everybody brought mass to the party! Rearranging gives

$$\omega = \sqrt{\frac{g}{\mu r}}$$

This is the minimum angular speed the cylinder must rotate for the person to be supported by friction. At or above this angular speed, the floor can be removed and the people will be pinned against the wall.

Extension Problems II – Friction & Circular Motion

1. Consider a car travelling along a circular path on a banked turn as shown in the diagram below.



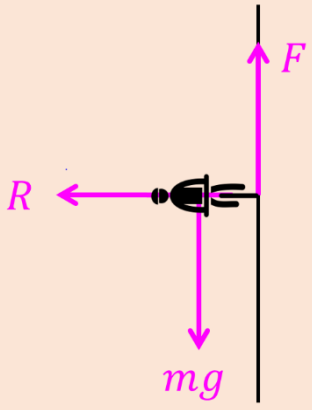
By modelling the car as a particle, show that the maximum speed it can take a circular corner of radius r, banked at an angle θ and with coefficient of friction between tyre and road of μ is given by

$$v_{\max} = \sqrt{gr\,\frac{\tan\theta + \mu}{1 - \mu\tan\theta}}$$

2. Show that minimum speed before the car starts to slip down the slope in the above problem is

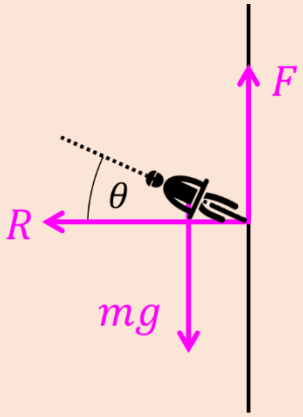
$$v_{\min} = \sqrt{gr\,\frac{\tan\theta - \mu}{1 + \mu\tan\theta}}$$

3. You may have noticed a slight problem with our treatment of The Wall of Death in Example 6.4 – if the rider is horizontal then surely the weight and friction act as a couple which would cause the rider to rotate anti-clockwise and fall off the wall?



This is a common simplification/error. The correct solution requires the rider to be at a slight angle to the horizontal, as shown in the diagram.

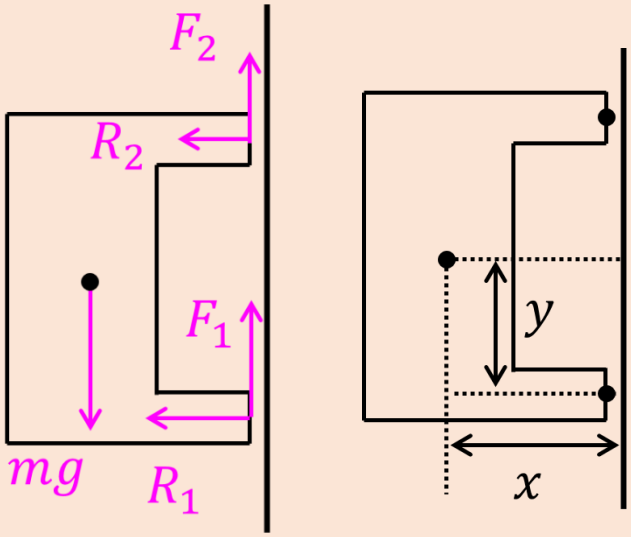
Consider a Wall of Death with a diameter of 12 m. We will assume that the centre of mas of the motorcycle and rider is 1 m above the road.



a. Calculate the minimum speed when the coefficient of friction is 0.5.

b. Calculate the angle of the motorcycle to the horizontal.

4. Now let's consider a car driving in a horizontal circle of a Wall of Death track with diameter 12 m. The centre of mass of the car is 1 m away from the track and the distance between the left and right wheels is 2 m. The diagrams below show the forces and dimensions.



a. Calculate the minimum speed when the coefficient of static friction is 1.

b. Take moments around the centre of mass to show that the car does not topple over at this speed

11. *Non-Inertial Frames & Fictious Forces

We touched on how the centrifugal force is an example of a fictitious (not real!) force that occurs when we try and apply Newton's 1st Law in non-inertial (accelerating) reference frames, such as a rotating reference frame. (Section 4.1)

Here's an excellent video that further investigates these forces. It starts with basic linear examples and then looks at rotating reference frames. It is without doubt the greatest intro to a physics video ever.

https://www.youtube.com/watch?v=bJMYoj4hHqU

12. *Moment of Inertia

The moment of inertia of an object is a measure of its resistance to changes in angular speed. It is analogous to mass, which is the resistance of an object to changes in *linear* speed. Below is a link to an interesting experiment about moments of inertia.

https://www.youtube.com/watch?v=cB8GNQuyMPc

The intermediate axis theorem blew my mind in 2nd year of university and I still don't understand it now! Below is a Veritasium video that explains it – very cool indeed.

https://www.youtube.com/watch?v=1VPfZ_XzisU

13. * Angular Momentum

Gyroscopic precession is pretty mind blowing. Here's another link from Veritasium.

https://www.youtube.com/watch?v=ty9QSiVC2g0