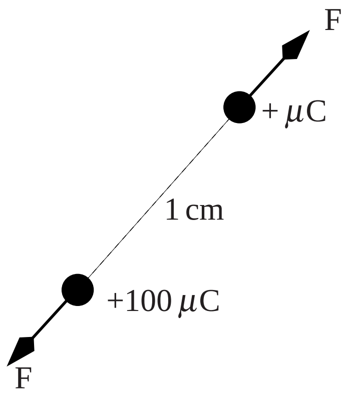
Tutorial Questions 1 – Electrostatics

1. Coulomb's Law gives the force per unit charge due to the interaction between stationary charges.
   (a) Using Coulomb's Law, determine the force on a 1 μC charge due to the electric field from a 100 μC charge when they are separated by 1 cm in air.
   (b) By what percentage is this force reduced when the air is replaced by a material with a relative permittivity of 3.2?
   (c) What mass would the 1 μC charge need to have to make the gravitational force at the surface of the Earth equal to the electric force on it due to the 100 μC charge described in part (a).
2. A square of side length 10 cm has four charges placed it its corners. Clockwise from the top right as viewed from above these are 5, 10, 6 and 3 nC. Determine the resultant electric field, including both magnitude and direction at the centre of the square due to this arrangement. Assume the medium is air. Find the direction relative to the diagonal line joining the 3 and 10 nC charges.
3. Use Gauss' Law to derive the equation for the electrc field a distance r from an infinite straight line of charge with density of λ C/m. If the charge density is 5 nC/m, determine the magnitude of the electric field at a point which is 4 cm from the line of charge. Consider the line of charge to be in air. What is the force (remembering force is a vector quantity) on a charge of -6 pC placed at this point.
4. For a system, two points, x and y, are along orthogonal directions from a 60 μC charge. For each of the following values of x and y, determine the potential difference (voltage) between them.
   (a) $x = 10, y = 15,$
   (b) $x = 10, y = 10,$
   (c) $x = 15, y = 10,$

   where distances are in cm. Explain why you could anticipate the value for (b) by symmetry, and relate this to the equipotential surfaces for a point charge. You may assume vaccuum conditions.
5. Determine the electric field between the plates of a charged parallel plate capacitor with a charge of magnitude 50 pC on each plate. Both plates are of an area of 10 mm^2 and the dielectric material between the plates has a relative permittivity of 2.5. The distance between the plates is 0.5 mm. Furthermore, calculate the voltage (potential difference) between the plates and the energy stored in the capacitor.

Solutions

1. The arrangement in the problem is as follows:



Coulomb's Law states that the force between two charges, q_1 and q_2 separated by a vector $\vec{r}$ has a magnitude of

$$|F| = \frac{q_1 q_2}{4\pi\varepsilon r^2}.$$

For vacuum, ε, the permitivity of the space between the two charges, is denoted ε_0 and has a value of approximately 8.854×10^{-12} F/m. The permittivity of air is very close to that of vaccuum, and the difference of about 0.5% is typically neglected.

(a) In the problem, $q_1 = 1 \times 10^{-6}$ C, $q_2 = 1 \times 10^{-4}$ C and $r = 1 \times 10^{-2}$ m, so that the magnitude of the force evaluates to <u>8.99×10^3 N</u> using the approximation $\varepsilon = \varepsilon_0$, and <u>$8.94 \times 10^3$ N</u> for $\varepsilon_r = 1.005$.

(b) The magnitude of the force can be written as

$$|F| = \frac{1}{\varepsilon_r} \frac{q_1 q_2}{4\pi\varepsilon_0 r^2},$$

so introducing a medium scales the force by a factor of $1/\varepsilon_r$. For a medium with relative permitivitty of 3.2, this amounts to a factor of ~ 0.313 or 31.3%, and the force is reduced by this scaling factor. The reduction can be expressed as a fraction of $1 - 1/\varepsilon_r$, being about 69%.

Alternatively, the reduced force can be readily calculated using the Coulomb formula, yielding 2.81×10^3N. Then the reduction in the force is $(8.99 - 2.81) \times 10^3 = 6.18 \times 10^3$N, which is a reduction of

$$\frac{6.18}{8.99} \times 100\% = 69\%.$$

(c) If we now imagine that this force is applied upwards with respect to the ground, then it would be opposed by the force of gravity. They would balance at a point when the mass of the charged object was such that

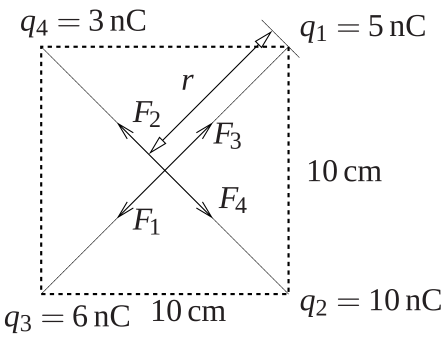
$$mg = 8.99 \times 10^3 \text{ N},$$

neglecting the dielectric difference between air and vaccuum. Since $g \approx 9.8 \mathrm{m\,s}^{-2}$,

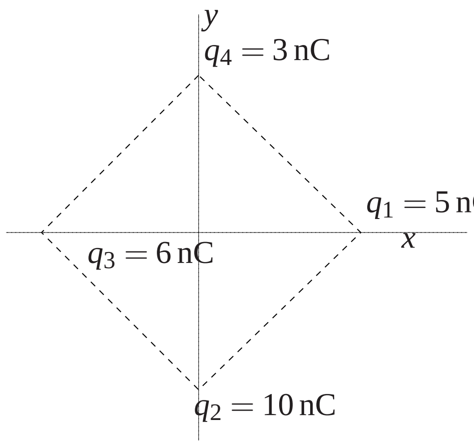
$$m = \frac{8.99 \times 10^3}{9.8} = 917\,\mathrm{kg}.$$

If we include the factor for air ($\varepsilon_r = 1.005$) then this is reduced to 913 kg.

2. To find the electric field at the centre of the square, we calculate the electric field due to each charge and sum them according to the principle of superposition. The arrangement in the problem is as follows:



In this case, we can simplify the picture by orienting the square relative the x and y such that they coincide with the diagonals. Placing the 5 and 6 nC charges along the positive and negative x-directions, we can now evaluate the forces.



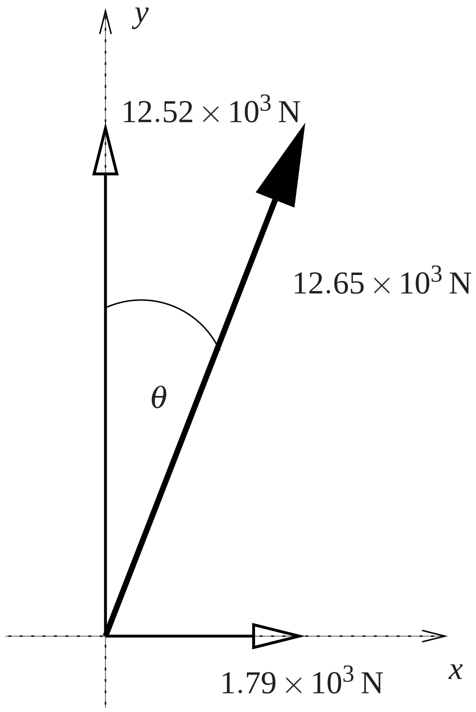
All charges are positive, so the electric fields *originate* from the corners. Thus the field from the first charge points in the negative x-direction, and has magnitude of

$$\frac{5 \times 10^{-9}}{4\pi\varepsilon r^2}\mathrm{N/C},$$

where r is the distance of the centre of the square from the corner. r is half the body diagonal of the square, and using pythaogoras we obtain $r = \frac{1}{2}\sqrt{10^2 + 10^2} = 5\sqrt{2}\,\mathrm{cm} \approx 7.07 \times 10^{-2}\,\mathrm{m}$. So the first field is 8.94×10^3N/C in the negative x-direction.

Similarly, the fields due to the three other charges are, proceeding in a clockwise direction from the first charge, 1.789×10^4N/C in the positive y-direction, 1.073×10^4N/C in the positive x-direction and 5.37×10^3N/C in the negative y-direction, respectively.

We can add these to get the total field at centre: $1.073 \times 10^4 - 8.94 \times 10^3$ N/C in the positive x-direction, and $-5.37 \times 10^3 + 1.789 \times 10^4$ N/C in the positive y-direction. In the normal sense of expressing cartesian vectors, the electric field is numerically expressable as $\vec{E} = 1.79 \times 10^3 \hat{\imath} + 1.252 \times 10^4 \hat{\jmath}$ in units of N/C. The magnitude of the field is 1.265×10^4 N/C.



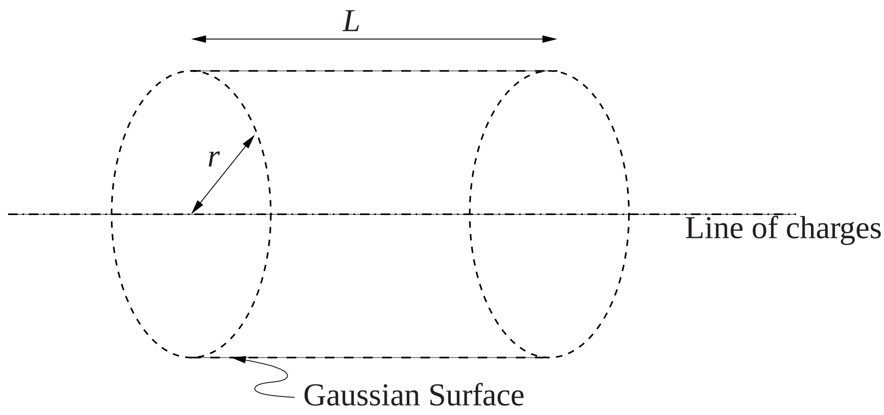
The diagonal between the 3 and 10 nC charges is the y-axis in the reorientated view. The angle that $\vec{E}$ makes with the y-axis can be obtained using the scalar product, $\vec{a}.\vec{b} = ab\cos(\theta)$. In this case $\vec{a} = \vec{E}$ and $\vec{b} = \hat{\jmath}$, so

$$+1.252 \times 10^4 = 1.265 \times 10^4 \times 1 \times \cos(\theta),$$

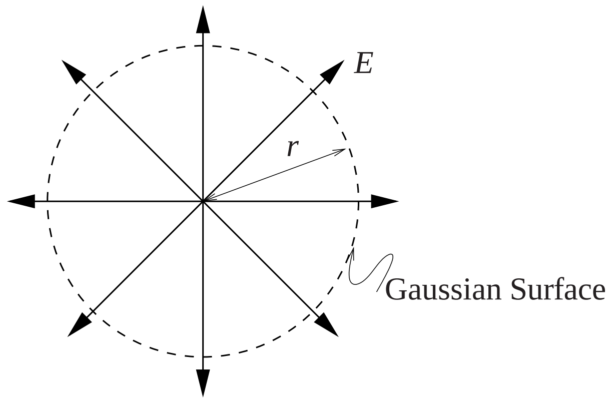
so $\theta \sim 8^\circ$.

Note, if the approximation of $\varepsilon = \varepsilon_0$ is used, this increases the field by a small amount to $\vec{E} = 1.80 \times 10^3 \hat{\imath} + 1.258 \times 10^4 \hat{\jmath}$ in units of N/C. The magnitude of the field is 1.271×10^4 N/C. The directions are not affected.

3. The main choice in the application of Gauss's Law is the geometry of the Gaussian surface. In practice we choose a surface to make the evaluation as simple as possible. In the case of a line of charge the simplest Gaussian surface we can use is a cylinder, coaxial with the line of charge.



By symmetry, the electric field must be perpendicular to the line of charge, and hence to the surface of the cylinder, and of constant value. By the same argument, the field is parallel to the ends of the cylinder.



Gauss's Law is expressed as $\phi = \int_S E dA = Q/\varepsilon$, where ϕ is the electric field flux through the surface. The integral can be divided into a part over the ends, and a part over the curved surface. The first part is zero as the electric field is parallel to the ends and therefore no flux passes through the ends. What remains is

$$\int_S E dA = \int_{\text{curved surface}} E dA = E \int_{\text{curved surface}} dA = E.2\pi r L = \frac{Q}{\varepsilon},$$

since E is a constant over the surface.

The charge contained, Q is λL, where λ is the line charge density. Hence,

$$E.2\pi r L = \frac{\lambda L}{\varepsilon},$$

and then

$$E = \frac{\lambda}{2\pi r \varepsilon}.$$

In words, the electric field from an infinite line of charge decreases as $1/r$, in contrast to the $1/r^2$ of a point charge.

If the charge density is 5 nC/m, at a distance of 4 cm from the line of charge the electric field is

$$E = \frac{5 \times 10^{-9}}{2\pi \times 4 \times 10^{-2} \times 8.854 \times 10^{-12} \times 1.005} = 2.24 \times 10^{3}\text{N/C},$$

radially outwards. If a point charge is placed 4 cm from the line of charge, it will experience a force $q\vec{E}$. For $q = -6 \times 10^{-12}$ C, the force is $2.24 \times 10^{3} \times (-6) \times 10^{-12} = -1.35 \times 10^{-8}$ N, the negative sign indicating that the force is directed toward the line of charge.

4. We have a point charge at the origin of some cartesian space. The voltage difference between two points in space is the difference in potential energy per unit charge between those two points. The potential energy change is

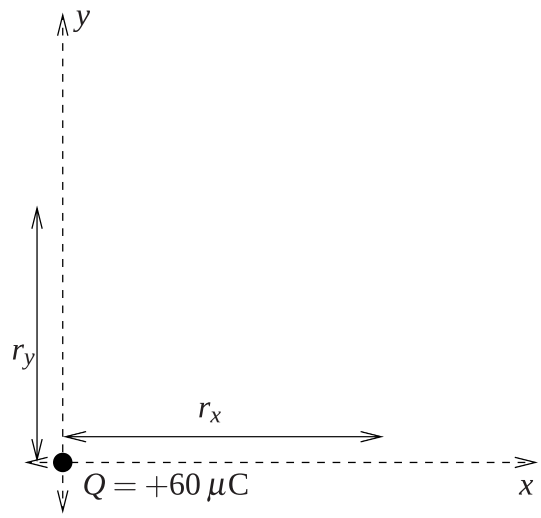
$$V = -\int_A^B E dr,$$

which is the energy per unit charge moving from point A to point B, and E is the electric field. By Coulomb's Law,

$$E = \frac{Q}{4\pi\varepsilon r^2},$$

so the voltage is

$$\begin{aligned} V &= -\int_A^B \frac{Q}{4\pi\varepsilon r^2} dr, \\ &= -\frac{Q}{4\pi\varepsilon}\int_A^B \frac{1}{r^2} dr, \\ &= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_A^B, \\ &= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]. \end{aligned}$$



The voltage difference between (10,0) and (0,15) (units of cm) due to a 60 μC charge at (0,0) is then simply

$$V = \frac{60 \times 10^{-6}}{4\pi 8.854 \times 10^{-12}}\left[\frac{1}{0.1} - \frac{1}{0.15}\right] = 1.80 \times 10^{6}\text{Volts}.$$

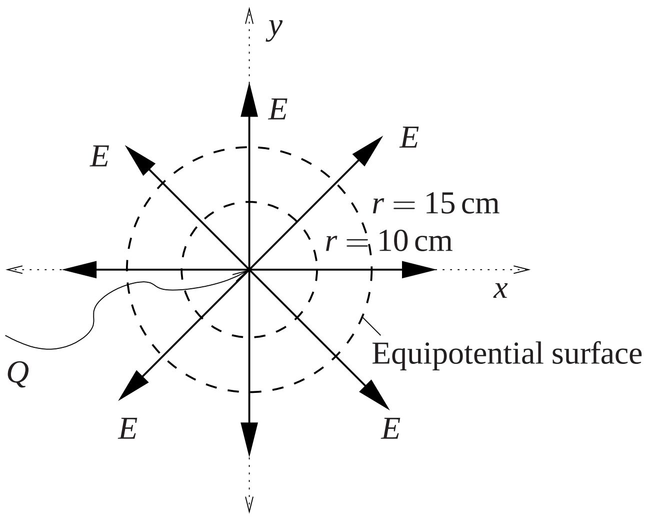
Similarly, the voltage difference between (10,0) and (0,10) (units of cm) due to a 60 μC charge at (0,0) is

$$V = \frac{60\times10^{-6}}{4\pi 8.854\times10^{-12}}\left[\frac{1}{0.1}-\frac{1}{0.1}\right] = 0\text{Volts},$$

and the voltage difference between (15,0) and (0,10) (units of cm) due to a 60 μC charge at (0,0) is

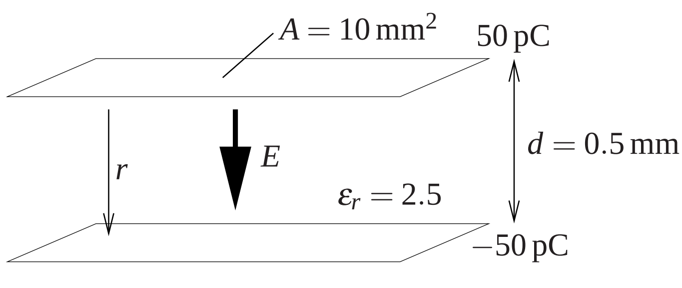
$$V = \frac{60\times10^{-6}}{4\pi 8.854\times10^{-12}}\left[\frac{1}{0.15}-\frac{1}{0.1}\right] = -1.80\times10^{6}\text{Volts}.$$

A point charge gives rise to spherically symmetric electric fields, and therefore spherically symmetric electrostatic potentials. In the case of the voltage between (10,0) and (0,10), the points lie on a sphere of radius 10 cm, and are therefore points of equal potential, so there is no voltage between these points. Indeed, for any two points on the surface of a sphere centred on the charge there is no potential difference. Such surfaces are the equipotentials for a single point charge.



5. For the parallel plate capacitor, in the approximation derived from *infinite planes of charge*, the electric field is uniform across the plates, and constant in value between them, given by the expression $E=\sigma/\varepsilon$, where $\sigma = Q/A$ is the magnitude of the charge density on each plate of area A. So, negelecting edge effects, the electric field in the capacitor is

$$E = \frac{\sigma}{\varepsilon} = \frac{Q}{A\varepsilon_0\varepsilon_r} = \frac{50\times10^{-12}}{10^{-5}\times8.854\times10^{-12}\times2.5} = 2.26\times10^{5}\ \text{V/m}.$$



The *voltage difference* between the plates is given by the integral of the electric field between the plates:

$$V = \int_0^d E dr = E \int_0^d dr = Ed,$$

so the voltage in the question is $2.26 \times 10^5 \times 0.5 \times 10^{-3} = 113$V.

In the lectures, it was shown that the total energy stored is simply $QV/2$, so in this simple capacitor, the stored energy is

$$\frac{50 \times 10^{-12} \times 113}{2} = 2.83 \times 10^{-9}\text{J}.$$